Chern-Simons, CMI. *

1 Recap on connections

Let M be a manifold. Instead of working in the general setup of principal bundles we prefer to consider vector bundles equiped with some extra geometric structure. For example the following categories (groupoids to be precise) are equivalent:

1. $\mathrm{GL}_n(\mathbb{R})$-bundles and $\mathrm{rk} = n$ real vector bundles.
2. $\mathrm{GL}_n(\mathbb{C})$-bundles and $\mathrm{rk} = n$ complex vector bundles.
3. $U_n(\mathbb{C})$-bundles and $\mathrm{rk} = n$ complex vector bundles with hermitian metric.
4. $\mathrm{SU}_n(\mathbb{C})$-bundles and $\mathrm{rk} = n$ complex vector bundles E with hermitian metric h and a section $s \in \Lambda^n E$ such that $h(s, s) = 1$.

We denote by $\Lambda^1_M = T^*_M$ the cotangent bundle and $\Lambda^i_M = \Lambda^i(T^*_M)$ the bundle of differential i-forms. The tangent bundle T_M will be denoted by Λ^{-1}_M as well.

Assume E is a complex vector bundle. We will denote by $E(U)$ the set of sections of E over an open $U \subset M$. It is convenient to think of E in terms of its sections, so we will omit U from the notation.

The following notion allows to speak about derivatives of sections of E along tangent vectors T_M.

Definition 1.0.1. A *connection* ∇ on E is a map

$$\nabla\colon E(U) \to \Lambda^1 \otimes_{\mathbb{R}} E\ (U)$$

satisfying the Leybniz rule $\nabla_v(fs) = v(f)s + f\nabla_v(s)$ for all $v \in T_M$ and $f \in \Lambda^0_M$.

Remark 1.0.2. It is important that the connection on E automatically provides a connection on any natural bundle associated with E: $E^*, \mathrm{End}(E), \Lambda^i E$, etc. This corresponds to the statement that any Lie algebra representation acts on its tensors. For example ∇ acts on the dual E^*, $\mathrm{End}(E) \simeq E^* \otimes E$, etc. The rule of thumb is that tensor products and natural pairings are with respect to the connections. For instance if $\phi \in \mathrm{End}(E)(U)$, then $(\nabla\phi)(s) = \nabla(\phi(s)) - \phi(\nabla s)$.

Note that the difference of any two connections $\nabla' - \nabla$ is Λ^0_M-linear, i.e. $\nabla' - \nabla$ is equal to some $A \in \Lambda^1_M \otimes \mathrm{End}(E)$. Thus the space

$$\mathcal{C}(E) = \{\text{a connection } \nabla \text{ on } E\}$$

is an *affine space* over $\Lambda^1_M \otimes \mathrm{End}(E)\ (M)$: the latter acts via $\nabla \rightsquigarrow \nabla + A$. In particular $T_{\nabla\in\mathcal{C}} \simeq \Lambda^1_M \otimes \mathrm{End}(E)(M)$ for all ∇.

Locally E is trivial, after fixing basis sections we get an identification $E \simeq \oplus_{i\leq k}\Lambda^0_M$: any connection (locally) has the form $\nabla = d + A$, where d is $\Lambda^0_M \to \Lambda^1_M$ and $A \in \Lambda^1_M \otimes \mathrm{End}(E)$.

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Exercise Assume M is a line bundle with connection ∇. Suppose L is N-th root of M, i.e. an isomorphism $L^{\otimes N} \simeq M$ is fixed. Show that L admits a natural connection induced by ∇ as follows. Choose non vanishing $s \in L(U)$, then $\nabla(s^{\otimes N}) = \nu \cdot s^{\otimes N}$ for some $\nu \in \Lambda^1$. Set $\nabla^{1/N}(s) = \frac{1}{N}\nu \cdot s$. Show that $\nabla^{1/N}$ doesn't depend on s and defines a connection on L.

Exercise Given the connection ∇ and any smooth path $\gamma\colon [0,1] \to M$, for each $t \in [0,1]$ there is a linear map $\Pi_t\colon E_{\gamma(0)} \to E_{\gamma(t)}$ called the *parallel transport* along γ. Define a connection $\gamma^*(\nabla)$ on $\gamma^*(E)$ over the interval and show that for each $v \in \gamma^*(E)_0$ there is a unique section $\tilde{v} \in \gamma^*(E)([0,1])$ such that $\gamma^*(\nabla)\tilde{v} = 0$. By the definition $\Pi_t(v) = \tilde{v}_t$.

Assume that in some gauge $\nabla = d + A$, then $\gamma^*(\nabla) = A(\dot{\gamma})dt$. Check that the matrix of $\Pi_{\nabla,\gamma}$ in the gauge is equal to the time-ordered exponent $\mathrm{Pexp}(-A(\dot{\gamma}))$, given by the iterated integral formula

$$\mathrm{Pexp}(X(t)) = \mathrm{id} + \int_0^t X(t_0)dt_0 + \int_0^t X(t_0)dt_0 \int_0^{t_0} X(t_1)dt_1 + \dots$$

satisfies $\frac{d}{dt}\mathrm{Pexp}(X(t)) = \dot{X}\,\mathrm{Pexp}(X(t))$.

The gauge group $\mathrm{Aut}(E)$ acts on $\Lambda^1_M \otimes \mathrm{End}(E)$ and $\mathcal{C}(E)$ differently. For $g \in \mathrm{Aut}(E)$ we have $g.\nabla = g \circ \nabla \circ g^{-1}$. If $\nabla = d + A$, then $g(d+A)g^{-1} = d + gdg^{-1} + gAg^{-1}$. Roughly speaking, the transformation of a 1-form A, viewed as a *connection*, acquire the term gdg^{-1}.

Exercise Fix a Lie group G (e.g. $\mathrm{GL}_n(\mathbb{C})$) with the Lie algebra $\mathfrak{g} \simeq T_{e\in G}$. The gauge transformation g is a map $g\colon M \to G$. Check that

$$-gdg^{-1} = g^{-1}dg = g^*(\mu),$$

where $\mu \in \Lambda^1_G \otimes \mathfrak{g}$ is the Maurer-Cartan form, i.e. the unique left invariant form such that $\mu_e(v) = v$ for any $v \in T_{e\in G}$. Show that μ satisfy the Maurer-Cartan equation:

$$d\mu + \frac{1}{2}[\mu,\mu] = d\mu + \mu \wedge \mu = 0.$$

Recall that the de Rham differential

$$d\colon \Lambda^i_M(U) \to \Lambda^{i+1}_M(U)$$

is the unique extension of the map $\Lambda^0_M(U) \to \Lambda^1_M(U)$ which satisfies the super-commutative Leibniz rule:

$$d(a \wedge b) = da \wedge b + (-1)^{|a|} a \wedge db.$$

Definition 1.0.3. The *curvature* of the connection $\nabla = d + A$ is 2-form

$$F^{\nabla} = d(A) + A \wedge A\colon \Lambda^2_M \otimes \mathrm{End}(E).$$

Exercise Check that 2-form F^{∇} is independent of coordiantes: $F^{g.\nabla} = gF^{\nabla}g^{-1}$. Check that the following formula makes sense

$$F^{\nabla} = \frac{1}{2}[\nabla,\nabla] \in E \to \Lambda^2_M \otimes E.$$

Deduce Bianchi's identity:

$$\nabla(F^{\nabla}) = [\nabla, [\nabla, \nabla]] = 0 \in \Lambda^3_M \otimes \mathrm{End}(E).$$

By the definition, a hermitian form $h(-,-)$ on E satisfy $h(ta, b) = h(a, \bar{t}b) = th(a, b)$ and $h(a, b) = \overline{h(b, a)}$ for all $a, b \in E(M)$ and $t \in \mathbb{C}$.

Exercise Show that any complex vector bundle E admits a hermitian form $h(-,-)$.

The $\mathrm{U}_n(\mathbb{C})$-bundle corresponding to E and an hermitian form $h(-,-)$ is denoted by (E, h). Similarly, $\mathrm{SU}_n(\mathbb{C})$-bundle is a triple (E, h, s), where $s \in \Lambda^n_{\mathbb{C}} E$ such that $h(s, s) = 1$.

Exercise Check that the complex vector bundle E lifts to a $\mathrm{SU}_n(\mathbb{C})$-bundle, if and only if, $c_1(E) = 0 \in H^2(M; \mathbb{Z})$. [1]

Let $\mathcal{C}(E, h) \subset \mathcal{C}(E)$ denotes the set of all *unitary* connections ∇ satisfying

$$dh(a, b) = h(\nabla a, b) + h(a, \nabla b).$$

Similarly $\mathcal{C}(E, h, s) \subset \mathcal{C}(E, h)$ we be the set of *special unitary* connections ∇ satisfying $\nabla(s) = 0$.
One can see that $\mathcal{C}(E, h)$ is an affine space over $\Lambda^1_M \otimes \mathfrak{u}(E, h)$ and $\mathcal{C}(E, h, s)$ is an affine space over $\Lambda^1_M \otimes \mathfrak{su}(E, h, s)$, where $\mathfrak{u}(E, h)$ is the Lie algebra of the unitary group $\mathrm{Aut}(E, h)$ and $\mathfrak{su}(E, h, s)$ is the Lie algebra of the special unitary group $\mathrm{Aut}(E, h, s)$. Note that the curvature F^{∇} always takes values in the corresponding Lie algebras.

2 Chern-Weil theory

Assume E is a complex vector bundle over some manifold M. Suppose that ∇ is a connection on E and $F(\nabla)$ its curvature.

Exercise Consider variation $\nabla \rightsquigarrow \nabla + \delta A$. Check that $\delta F(\nabla) = \nabla(\delta A)$. Use Bianchi identity to show that $d\,\mathrm{tr}(F(\nabla)^k) = 0$ and

$$\delta\,\mathrm{tr}(F(\nabla)^k) = kd(\mathrm{tr}(F(\nabla)^{k-1}\delta A)$$

for all k. Deduce that $[\mathrm{tr}(F(\nabla)^k)] \in H^{2k}(M; \mathbb{C})$ is well-defined and independent of ∇.

Recall that Newton's polynomials $\sum_{k \geq 1} t^k \,\mathrm{tr}(F^k)$ are expressed in terms of characteristic polyomial $\det(\mathrm{id} + tF)$, and vice versa.

Corollary 2.0.1. *We have*

$$[\det(\mathrm{id} - \frac{1}{2\pi i} F(\nabla))] = c(E) \in H^*(M; \mathbb{Z}),$$

where $c(E) = 1 + c_1(E) + c_2(E) + \dots,$ *is the Chern polynomial.*

[1] There are many ways to define $c_1(E)$, see below. You can use the fact that $c_1(E) = 0 \iff \det E = \Lambda^{\mathrm{rk}\, E} E$ admits smooth non vanishing section.

Proof. Both parts are multiplicative in E. Use the splitting principle to reduce to the case E is a line bundle. The classifying space for line bundles is $\mathbb{C}P^\infty$ with the universal line bundle $\mathcal{O}(-1)$. On the other hand $c_1(\mathcal{O}(-1))$ is detected by its restriction to the submanifold $\mathbb{C}P^1$. The only issue is to determine the coefficient before F in the case of the line bundle $\mathcal{O}(-1)$ with, say, Chern connection over $\mathbb{C}P^1$.

In the restriction to $\mathbb{C} \subset \mathbb{C}P^1$, the curvature of the Chern connection on $\mathcal{O}(-1)$ is equal to $F = \frac{d\bar{z}dz}{(1+|z|^2)^2}$ (in the restriction to $\mathbb{C} \subset \mathbb{C}P^1$). Hence $\int_{\mathbb{C}P^1} F = 2\pi i$, on the other hand $c_1(\mathcal{O}(-1)) = -1$, so $c_1 = 1 - \frac{1}{2\pi i}F$. □

3 Chern-Simons action

Let M be a compact manifold of $\dim M = 3$ with possibly non empty boundary $\Sigma = \partial M$. We assume that M is oriented, i.e. one can define the integral $\int_M v_3$ for any top-form $v_3 \in \Lambda^3(M)$.

It turns out that the above section considerations simplify because $\mathrm{SU}(N)$-bundles over M and ∂M are in fact trivial.

Proposition 3.0.1. *Any $\mathrm{SU}(N)$-bundle E on M is trivial.*

Proof. Recall that the isomorphism classes of $\mathrm{SU}(N)$-bundles on M are in 1-1 correspondence with the set of maps $[M, B\,\mathrm{SU}(N)]$ up to homotopy. It is well-known that $\pi_{<3}(\mathrm{SU}(N)) = 0$, hence $\pi_{<4}(B\,\mathrm{SU}(N)) = 0$ and so $[M, B\,\mathrm{SU}(N)] = 0$ for any 3-dimensional cellular complex M. Thus E is isomorphic to the trivial one. □

To complete the discussion we sketch the following topological result.

Theorem 3.0.2 (Rokhlin)**.** *The tangent bundle T_M is trivial.*

Proof. Passing to the double construction $M \cup_{\partial M} \bar{M}$, one can assume that $\partial M = \emptyset$. here we abuse notation and denote by $[M]$ the reduction $[M] \in H^3(M; \mathbb{Z}/2)$ modulo 2.

Denote by $w_i \in H^i(M; \mathbb{Z}/2), i = 1, 2, 3$ the Stiefel-Whitney classes of T_M. Recall that $w_1 = 0$ iff M is orientable and $w_3 = \chi(M) \cdot [M]$ is the Euler characteristics times the fundamental class of M modulo 2. Hence $w_1, w_2 = 0$.

1. Let $w = 1 + w_1 + w_2 + w_3 \in H^*(M; \mathbb{Z}/2)$. For any $x \in H^*(M; \mathbb{Z}/2)$ we denote by $\int_{[M]} x \in \mathbb{Z}/2$ the pairing of x with $[M] \in H^3(M; \mathbb{Z}/2)$. Recall the Wu's formula:

$$\int_{[M]} \mathrm{Sq}(x) = \int_{[M]} x\,\mathrm{Sq}^{-1}(w),$$

where $\mathrm{Sq}(x) = \sum_{i\geq 0} \mathrm{Sq}^i(x)$ is the authomorphism of $H^*(M; \mathbb{Z}/2)$ given by the full Steenrod square and Sq^{-1} is its inverse. In our case $\mathrm{Sq}^{-1}(w) = \mathrm{Sq}^{-1}(1 + w_2) = 1 + w_2$, hence

$$\int_{[M]} w_2 \cup x = \int_{[M]} \mathrm{Sq}^2(x_1)$$

for any $x_1 \in H^1(M; \mathbb{Z}/2)$. Since $\mathrm{Sq}^{>1}(x_1) = 0$ and Poincaré pairing is not degenerate we conclude that $w_2 = 0$.

2. It is well-known that $w_2 = 0$ implies that T_M admits a spin structure. Recall that we have a double covering $\mathrm{SU}(2) \simeq \mathrm{Spin}(3) \to \mathrm{SO}(3)$ and the spin structure means T_M is a reduction of some $\mathrm{SU}(2)$-bundle such that its reduction to $\mathrm{SO}(3)$ is equivalent (as real vector bundle) to T_M. By the above any $\mathrm{SU}(2)$-bundle on M is trivial, hence T_M is trivial as well. □

From general cobordism theory it follows that any oriented M with $\partial M = \emptyset$ is the boundary of a 4-dimensional oriented manifold N. Since E is trivial, one can extend it, as well as ∇, to a connection $\tilde{\nabla}$ on a trivial bundle over N.

In the following we will work differential forms with values in $\mathrm{Mat}_{n\times n}(\mathbb{C})$. Recall that tr is symmetric form on matrices, hence for any collection of forms $a_i \in \Lambda^{d_i} \otimes \mathrm{Mat}_{n,n}, i \leq k$ one can define a form $\mathrm{tr}(a_1 \wedge \ldots \wedge a_k) \in \Lambda^d$, where $d = \sum_i d_i$. The usual symmetrices of tr work in this graded commutative setting as well.

Exercise Check that $\mathrm{tr}(a_1 \wedge \ldots \wedge a_k) = (-1)^{d'} \mathrm{tr}(a_2 \wedge a_3 \wedge \ldots \wedge a_k \wedge a_1)$, $d' = d_1(d_2 + d_3 + \ldots + d_k)$ is equal to the number of "moves" of a_1 through $a_2, \ldots, a_k$.

Exercise Show that for any $\mathrm{SU}(n)$-bundle $\tilde{E}$ on 4-dimensional N with a connection $\tilde{\nabla}$ and its curvature F, we have

$$\frac{1}{4\pi^2} \int_N \mathrm{tr}(F \wedge F) = \int_{[N]} c_2(E).$$

If $\partial N = M$, then

$$\frac{1}{4\pi^2} \int_M \mathrm{tr}(F(\tilde{\nabla}) \wedge F(\tilde{\nabla})) \in \mathbb{R}/\mathbb{Z}$$

is well-defined and is independent of N and the extension $\tilde{\nabla}$.

If $\tilde{\nabla} = d + \tilde{A}$ in some trivialization of $\tilde{E}$, then one immediately checks that

$$d\,\mathrm{tr}(\tilde{A} \wedge d\tilde{A} + \frac{2}{3} \tilde{A} \wedge \tilde{A} \wedge \tilde{A}) = \mathrm{tr}(F(\tilde{\nabla}) \wedge F(\tilde{\nabla})).$$

Restriction of the trivialization to the boundary allows to write $\nabla = d + A$, we obtain that

$$\frac{1}{8\pi^2} \int_M \mathrm{tr}(AdA + \frac{2}{3} AAA) \in \mathbb{R}/\mathbb{Z},$$

is a well-defined invariant of M and A.

That the way how Chern and Simons came up to this functional. To be consistent with Witten's notation we define Chern-Simons functional by

$$S_{CS}(A) = \frac{1}{4\pi} \int_M \mathrm{tr}(AdA + \frac{2}{3} AAA) \in \mathbb{R}/2\pi\mathbb{Z}.$$

Thus, for closed M, $e^{ikS_{CS}(A)}$ is gauge invariant for any integer k. For convenience we will set $k' = \frac{k}{4\pi}$ and $S'_{CS}(A) = \int_M \mathrm{tr}(AdA + \frac{2}{3} AAA)$.

4 Gauge invariance of Chern-Simons

Here we will consider the case of 3-manifold M with boundary. Let $\mathcal{C} = \mathcal{C}(E, h, s)$ denotes the space of special unitary connections on trivial bundle E. Under the variation $A \rightsquigarrow A + \delta A$, we have:

$$\delta S'_{CS} = \int_{\partial M} \mathrm{tr}(\delta A\ A) + 2 \int_M \mathrm{tr}(F(\nabla)\ \delta A) = \nu_\nabla(\delta A) + 2\eta_\nabla(\delta A).$$

The formula defines two 1-forms $\nu, \eta \in \Lambda^1_{\mathcal{C}}$. Since ν depends only on the restriction to ∂M, we will consider the corresponding term $\nu^B \in \Lambda^1_{\mathcal{C}(\partial M)}$ as well.

If we go beyond 0-forms and denote the de Rham differential on the space $\mathcal{C}(M)$ by $\mathcal{D}$, then the above equality reads as $\mathcal{D}(\nu + 2\eta) = 0$. Using the usual formula for $\mathcal{D}$ we obtain that

$$\mathcal{D}\nu^B\ (\delta A, \delta B) = -\int_{\partial M} \mathrm{tr}(\delta A \wedge \delta B),$$

is equal to the symplectic form on $\mathcal{C}(\partial M, E, h)$, it is affine parallel.

The form η is clearly gauge invariant, while ν is not. Namely, since $g.\delta A = g\delta A g^{-1}$ and $g.A = gAg^{-1} + gdg^{-1}$ we obtain

$$(g^*\nu - \nu)(\delta A) = \int_{\partial M} \mathrm{tr}(g^{-1}dg\ \delta A) = \delta f_g,$$

where the functional $f_g(-) \in \Lambda^0_{\mathcal{C}}$ is given by

$$f_g(A) = \int_{\partial M} \mathrm{tr}(g^{-1}dg\ A). \tag{4.0.1}$$

We obtain

$$\delta(g^*S' - S') = (g^*\nu - \nu)(\delta A) = \delta f_g.$$

Hence for any A we have $S'(g.A) - S'(A) - f_g(A) = C_g$ for some C_g which depends only on g. To recover C_g it is enough to compare both parts at $A = 0$; this simple computation shows an equality for all A:

$$S'(g.A) - S'(A) = \int_{\partial M} \mathrm{tr}(g^{-1}dg \wedge A) + \frac{1}{3}\int_M \mathrm{tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg). \tag{4.0.2}$$

Let us take a closer look at the terms appearing on the right hand side.

1. The second term in (4.0.2):

$$S'_{top}(M, g) = \frac{1}{3}\int_M (g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg),$$

appears in the Witten's correction to Wess-Zumino sigma-model on ∂M and up to a normalization its exponent is determined by a restriction to ∂M. As we know $g^{-1}dg = g^*(\mu)$, where μ is the Maurer-Cartan form on $\mathrm{SU}(N)$. It is known that $\int_{\mathrm{SU}(2)} \mathrm{tr}(\mu \wedge \mu \wedge \mu) = 24\pi^2$ and hence the biinvariant form $w_3 = \frac{1}{24\pi^2}\mathrm{tr}(\mu\wedge\mu\wedge\mu) \in \Lambda^3(\mathrm{SU}(N))$ represents the generator of $H^3(\mathrm{SU}(N);\mathbb{Z})$. In particular, for closed M one has

$$S'_{top}(M, g) = 8\pi^2 \cdot \deg(g),$$

where $\deg(g) = \int_M w_3 \in \mathbb{Z}$. In the case of gauge group SU(2), $\deg(g)$ is the degree of the map $g\colon M \to \mathrm{SU}(2) = S^3$. This confirms that $e^{ik'S'_{CS}}, k \in \mathbb{Z}$ is gauge invariant if $\partial M = \emptyset$.

Lets go back to the general case $\partial M \neq \emptyset$. It follows that the value $e^{i\frac{k}{4\pi}S'_{top}(M,g)}$ depends only on the restriction $g|_{\partial M}\colon \partial M \to \mathrm{SU}(N)$.

A direct calculations shows that for any $g_1, g_2\colon M \to \mathrm{SU}(N)$ we have Polyakov–Wiegmann formula

$$S'_{top}(M, g_1g_2) - S'_{top}(M, g_1) - S'_{top}(M, g_2) = -\,\mathrm{PW}(g_1, g_2),$$

where we set

$$\mathrm{PW}(g_1, g_2) = \int_{\partial M} \mathrm{tr}(g_1^{-1}dg_1 \wedge dg_2g_2^{-1}).$$

2. The term $f_g(A)$ depends only on a restriction to ∂M and a simple calculation shows that

$$f_{g_1 g_2}(A) - f_{g_1}(g_2.A) - f_{g_2}(A) = \mathrm{PW}(g_1, g_2).$$

The discussion above implies that

$$e_g(A) = e^{\frac{i}{4\pi}(f_g(A) + S'_{top}(g))}$$

depends only on the restrictions to ∂M and satisfies the *cocycle condition*:

$$e_{g_1 g_2}(A) = e_{g_1}(g_2.A) \cdot e_{g_2}(A),$$

moreover we have

$$e^{ikS_{CS}(g.A)} = e^{ikS_{CS}(A)} \cdot e_g^k(A).$$

In other words $e_g(A)$ defines an equivariant structure on the trivial line bundle $\mathcal{C}(M, h) \times \mathbb{C}$ with respect to the action of $\mathrm{Maps}(M, \mathrm{SU}(N))$ via the identification of fibers at A and $g.A$, respectively:

$$e_g(A) \colon \mathbb{C}_A \to \mathbb{C}_{g.A}.$$

The cocycle $e_g(A)$ defines $\mathrm{U}(1)$-bundle $\tilde{\mathcal{L}}$ on the quotient $\mathcal{C}(M)/\operatorname{Maps}(M, \mathrm{SU}(N))$, such that $e^{ikS_{CS}(A)}$ is a unitary section of $\tilde{\mathcal{L}}^{\otimes k}$. Since $e_g(A)$ depends only on the restriction of A and g to the boundary ∂, the same construction defines $\mathrm{U}(1)$-bundle $\mathcal{L}$ over $\mathcal{C}(\partial M)$ and the restriction map $p \colon \mathcal{C}(M) \to \mathcal{C}(\partial M)$ provides the identification $\tilde{\mathcal{L}} \simeq p^* \mathcal{L}$.

3. The standard approach to a quantization of gauge theory via path integrals is to first eliminate the redundancy provided by the gauge action and make sense of the integration over the space of orbits. In other words one has to pass from $\mathcal{C}(M)$ and the action $e^{ik' S'_{CS}}$ with values in the trivial line bundle, to the quotient $\mathcal{C}(M)/\operatorname{Maps}(M, \mathrm{SU}(N))$ with $e^{ik' S'_{CS}}$ viewed a section of the line bundle $\mathcal{L}$. **TODO: the cocycle for decorated Chern-Simons**

5 Equivariant connections

Assume X is a manifold provided with an action of Lie group G. Suppose E is G-equivariant bundle over X. In other words for each $x \in X$ and $g \in G$ we have an isomorphism $e_g(x) \colon E_x \to E_{g.x}$ satisfying the cocycle condition:

$$e_{g_1 g_2}(x) = e_{g_1}(g_2.x) e_{g_2}(x). \tag{5.0.1}$$

If the quotient X/G is well-defined, then E descends to a bundle over X/G. We would like to describe connections on E which come from X/G. Suppose ∇ is a connection on E. Then for any $g \in G$, the pullback of $g^*\nabla$ defines a connection on E via the identification $e_g \colon E \to g^*E$.

Proposition 5.0.2. *Suppose (E, ∇) locally takes the form $\nabla = d + \Phi$ for $\Phi \in \Lambda^1 \otimes \mathrm{Mat}_{n \times n}(\mathbb{C})$, so that we can write $e_g(x) \in \mathrm{GL}_n(\mathbb{C})$. Then*

$$g^*(\nabla) = d + g^*(e_g^{-1} de_g) + e_g^{-1} g^*(\Phi) e_g. \tag{5.0.3}$$

Proof. DIY. □

Remark 5.0.4. Here g^* denotes the usual pullback of differential forms. Note that this the action on the *right*. For the trivial action on X, $g^* = \mathrm{id}$ and the formula recovers the usual gauge transformation formula, where we worked with the left action convention.

Definition 5.0.5. A connection ∇ on the equivariant bundle E is *invariant* if $g^*\nabla = \nabla$.

It is useful to have infinitesimal characterization of equivariance. Locally in the first order in δg viewed as an element of $\mathfrak{g}$ we can write

$$e_{1+\delta g}(x) = \mathrm{id}_n + \mu_{\delta g} \in \mathrm{GL}_n(\mathbb{C}).$$

In the standard notation this defines $\mu_a(x) \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ for any $a \in \mathfrak{g}$.

The element μ_a has no invariant meaning, because it's definition involves fibers over different points. Using ∇ one can set

$$\alpha_a = \mu_a + \Phi(X_a),$$

then it is easy to check the formula defines a tensor $\alpha\colon \mathfrak{g} \to \mathrm{End}(E)$.

The variation of (5.0.3) with respect to $g = 1 + \delta g$, for $\delta g = a \in \mathfrak{g}$ gives:

$$X_a(\nabla) = d\mu_a + \mathcal{L}_{X_a}(\Phi) + [\Phi, \mu_a].$$

Substitution $\mu_a = \alpha_a - \Phi(X_a)$ provides an invariant expression

$$X_a(\nabla) = \nabla\alpha_a + X_a\lfloor F(\nabla) \in \Lambda^1 \otimes \mathrm{End}(E), \tag{5.0.6}$$

where $X_a\lfloor-$ is the contraction with the vector field X_a. The cocycle condition on e_g translates to

$$\mu_{[a,b]} = X_b.\mu_a - X_a.\mu_b + [\mu_a, \mu_b], \tag{5.0.7}$$

where X_a, X_b are vector fields on X induced by infinitesimal action of $a, b \in \mathfrak{g}$.

Remark 5.0.8. The cocycle condition on e_g is equivalent to the group homomorphism

$$f\colon G \to G \ltimes M,$$

which sends g to (g, e_g). Here $M = \mathrm{Maps}(X, \mathrm{GL}_n(\mathbb{C}))$ and the right action by G via pullback turns $G \ltimes M$ into a semidirect product. On the level of Lie algebras we get a map $df\colon \mathfrak{g} \to \mathfrak{g} \ltimes \mathfrak{m}$. Here $\mathfrak{m}$ is the Lie algebra of M provided with the action of G, that will be written infinitesimaly by $m.a$ for $m \in \mathfrak{m}$ and $a \in \mathfrak{g}$. One can show that the Lie algebra structure on $\mathfrak{g} \ltimes \mathfrak{m}$ is given by the formula $[(a, m), (a', m')] = ([a, a'], [m, m'] + m.a' - m'.a)$. The infinitesimal cocycle condition says that *df* is a map of Lie algebras. More generally it is equivalent to the Maurer-Cartan equation for $\mu \in \Lambda^1\mathfrak{g}^\vee \otimes \mathfrak{m}$ given by $d^{CE}f + \frac{1}{2}[f, f] = 0$, where d^{CE} is the Chevalley-Eilenberg differential.

This discussion also explains that *locally* in G in order to provide e_g satisfying (5.0.1) it is enough to find μ_a satisfying (5.0.7).

Invariantly we get

$$F(\nabla)(X_a, X_b) + \nabla_{X_b}\alpha_a - \nabla_{X_a}\alpha_b + [\alpha_a, \alpha_b] - \alpha_{[a,b]} = 0, \tag{5.0.9}$$

Remark 5.0.10. One can view α as an element in $\Lambda^*\mathfrak{g}^\vee[-1] \otimes \mathrm{End}(E)$ (i.e. $\mathfrak{g}^\vee$ is in degree 1). The latter has natural degree 1 operators given by the Chevalley-Eilenberg differential δ^{CE} acting on $\Lambda^*\mathfrak{g}^\vee$ and $m \circ \nabla$ induced by the composition

$$\mathrm{End}\, E \to \Lambda^1 \otimes \mathrm{End}(E) \to \mathfrak{g}^\vee \otimes \mathrm{End}\, E,$$

where the last arrow is the Lie algebra action on X. Our equation is equivalent to the condition that $d^{CE} + m \circ \nabla + \alpha$ is a differential.

Assume s is an invariant section of E, i.e. $s(g.x) = e_g(x)s(x)$. Infinitesimaly this is the condition $X_a(s) = \mu_a s$, or in invariant terms

$$\nabla_{X_a} s = \alpha_a s. \tag{5.0.11}$$

TODO: Complex structure on $\mathcal{C}(\partial M, h, s)$ and the induced Chern connection of $\mathcal{L}$. Show that the curvature $F(\mathcal{L})$ is the pullback from GIT quotient.

6 Path integrals in a nutshell

1. Fix a finite dimensional vector space V over reals together with an Euclidian metric $(-,-)$. The metric defines a volume form such that the unit cube has unit volume. We denote the corresponding measure by $\mathcal{D}a$. Assume $D\colon V \to V$ is a positive self-adjoint operator: $(Da, b) = (a, Db)$ for all $a, b \in V$ and $(Da, a) > 0$ for all $a \neq 0$.

We list three important corollaries of the identity $\int_{\mathbb{R}} e^{-\frac{1}{2}t^2} dt = \sqrt{2\pi}$. We will refer to elements of V as (classical) *fields* and call the function $S(a) = \frac{1}{2}(a, Da)$ the *action functional.* Let $k > 0$ be some parameter. We start with the *partition function*:

$$Z := \int_V e^{-kS(a)} \mathcal{D}a = \frac{1}{\sqrt{\det(kD/2\pi)}}. \tag{6.0.1}$$

One can introduce an auxiliary *source field* $f \in V$:

$$Z[f] := \int_V e^{-kS(a)+(f,a)} \mathcal{D}a = e^{\frac{1}{2}(f,(kD)^{-1}f)} \cdot Z. \tag{6.0.2}$$

Partial derivatives of $Z[f]$ can be used to compute *correlator functions.* For example

$$\langle (f,a),(g,a) \rangle := \frac{1}{Z}\int_V e^{-kS(a)}(f,a)(g,a)\mathcal{D}a = \frac{1}{Z}\delta_f \delta_g Z[f]|_{f=0} = (f,(kD)^{-1}g), \tag{6.0.3}$$

is the formula for *2-point correlator function.* In general, if $F(a)$ is any polynomial function in a we can define $\langle F(a) \rangle$. It is equal to the expectation value of $F(a)$ with respect to the probability measure $\det(kD/2\pi) \cdot e^{-\frac{1}{2}(a,kDa)} \mathcal{D}a$. An n-point function is computed by means of *Wick formula*:

$$\langle (f_1,a)\dots(f_n,a) \rangle = \sum_{\Gamma_n} \prod_{e \in E(\Gamma_n)} (f_{e_0}, (kD)^{-1} f_{e_1}),$$

where Γ_n is the set of graphs on labeled vertices $1,\dots,n$ with the set of edges $e \in E(\Gamma_n)$ such that none of different edges has an intersections. Here e_0, e_1 denotes unordered pair of end points of the edge e. Clearly n-point function vanish for odd n. One should think of $f_i \in V$ as a vertex labeled by f_i with an attached half edge. Wick formula says that one has sum over all the ways to glue half edges of different f_i, the resulting graph has vertices labeled by f_i and edges decorated by self-adjoint $(kD)^{-1}$.

The 2-point function determines all correlators, the inverse D^{-1} is called *propagator* and in practice often appear as a Green function.

Notice that if we forget about integrals in the middle, all formulas on the right hand side, by analytic continuation, make sense as long kD is non degenerate. For any complex number $k \neq 0$ and non degenerate D we *define* the integrals in the middle to be consistent with the formulas on the right hand sides. The assumption that V is finite dimensional was necessary only to define $\det(kD)$. One can try to overcome even this restriction by using *zeta regularization*: if $T_s := \mathrm{Tr}[(kD)^s]$ is well-defined for some open domain in $\mathbb{C}$, then one can set $\frac{d}{ds}T_s|_{s=0} = \det(kD)$.

In practice people take $k = \frac{i}{\hbar}$ and pretend that we are working in quantum field theory setting with Planck's parameter $\hbar$, or consider real $k > 0$ as the temperature in statistical mechanics.

So far we considered only the simplest case with the quadratic action $S(a)$. Usually the action includes, akin to the quadratic kinetic term, an *interactions* of fields. The general action can be written as $S_I(a) = S(a) + I(a)$, where $I(a) = \sum_i I_i(a)$ and $I_i \in \mathrm{Sym}^{d(i)}(V^*)$ for some degree function $d(i) > 2$. Here we allow different terms of the same degree. If we consider the term I as the *perturbation* of S, then we can relate theory with interaction to the correlators of quadratic theory:

$$\langle F(a)\rangle_I = \frac{1}{Z_I}\int e^{-kS(a)-kI(a)}F(a)\mathcal{D}a = \frac{1}{Z_I/Z}\sum_n \frac{(-k)^n}{n!}\langle I(a)^n F(a)\rangle,$$

and

$$\frac{Z_I}{Z} = \sum_n \frac{(-k)^n}{n!}\langle I(a)^n\rangle.$$

Notice that degree d part in $I(a)^n$ is a sum over monomials $\prod_{i\leq n} I_i(a)^{k_i}$ with $\sum k_i d(i) = d$ and $\sum k_i = n$. The value $k^n\langle I_{i_1}(a)\cdot I_{i_n}(a)\rangle$ is given by a sum over all graphs $\Gamma = \Gamma_{i_1,\dots,i_t}$ with n vertices labeled by the tensor $I_{i_t} \in \mathrm{Sym}^{d(i_t)}(V^*), t \leq n$ of valency $\deg I_{i_t}$. The number of edges of Γ is equal to $\frac{1}{2}\sum_t d(i_t)$. If we decorate each edge of $\Gamma = \Gamma_{i_1,\dots,i_t}$ with the propagator $(kD)^{-1}$ one can compute the contribution of the term corresponding to Γ, similarly to the Wick's formula: it is equal, up to a combinatorial coefficient, to $k^{n-\frac{1}{2}\sum_t d(i_t)}\gamma_{i_1,\dots,i_t}(D,I)$, where $\gamma(D)$ depend only on D and interactions I. Here we used the fact that each edge contribbutes to k^{-1} and each vertex I_{i_t} by $k^{d(i_t)}$. Note that $\chi(\Gamma) = n - \frac{1}{2}\sum_t d(i_t) < 0$ because $d(i_t) > 2$. The graphs are called Feynman's diagrams. They provide an expansion in powers of k^{-1}, such that the terms of order k^{χ} are given by graphs Γ with Euler characteristics equal to χ. Here Γ allow loops, but without vertices of valence $1, 2$, thus $\chi < 0$.

The similar considerations provide an expression for $\langle F(a)\rangle_I$ with $F \in \mathrm{Sym}^d(V)$, as a sum over Feynman diagrams with d exterior half edges, which should be couple with the star shaped vertex F with d half edges.

This is sketch of the perturbative analysis of QFT. One can justify the approach by using change of variables $a \rightsquigarrow \frac{a}{\sqrt{k}}$ and noting that it is equivalent to working with $k = 1$ and the interaction $I(a,k) = \sum k^{-d(i)/2}I_i(a)$. One can expect that for large enough k the contribution of the term $I(a,k)$ with arbitrary small coupling constants implies arbitrary small changes of correlators. But it is not the case. For instance the non perturbative partition function with the action $S(t) = \frac{1}{2}t^2 + k^{-3/2}t^3$ doesn't make any sense for any $k > 0$. Thus the power series described with Feynman technique k^{-1} are formal and they don't converge.

If $I(a)$ is written in terms of monomials (f,a) for various $f \in V$, then $\langle F(a)\rangle_I$ admits a nice combinatorial expansion in powers of k^{-1}, it is called Feynman's diagrams expansion. The main feature of this expansion is that its terms are parametrized by graphs with labeled vertices and their values are computed by means of propagator D^{-1}.

Let's go back to the quadratic theory and assume now that D is degenerate. The kernel $K = \ker D$ is the space of so called *zero modes*. Taking the orthogonal complement $K^{\perp}$ with respect to the inner product $(-,-)$, we obtain a decomposition $V = K \oplus K^{\perp}$. One can view the translation along K as a symmetry of the theory. All integrals of invariant expressions under the translation along K are equal to the integrals over $K^{\perp}$ times the volume of K.

There are two solutions to this problem. The first is to restrict our space of fields to $K^\perp$, where $D\colon K^\perp \to K^\perp$ is invertible. The second is to introduce a parameter m such that $D+m\cdot\mathrm{id}$ is invertible, solve theory there to get an analytic expression in m and then pass to the limit $m\to 0$. In fact the both approaches are equivalent, this is related to the fact that

$$\sqrt{m/2\pi}e^{-\frac{1}{2}mt^2} \to \delta_0(t),$$

as $m \to 0$.

7 Example: abelian Chern-Simons over S^3 and linking number of knots

We will apply the previous discussion to the Chern-Simons theory with gauge group is U(1) and $M = S^3$

A connection in U(1)-bundle has a form $d + iA$, where $A \in \Lambda^1(M)$, and the action is given by $S'_{CS}(A) = \int_M A \wedge dA$. Since $S^3 = \mathbb{R}^3 \cup \{\infty\}$ one can consider any 1-form on S^3 as a 1-form on $\mathbb{R}^3$ with sufficiently fast decay on infinity.

Recall that a metric g on M defines the Hodge star-operator

$$*\colon \Lambda^i_M \to \Lambda^{\dim M - i}_M,$$

defined by the equality $a \wedge *b = (a,b)\operatorname{Vol}_g$, where $(-,-)$ is the induced metric on Λ^i and Vol_g is the associated volume form. We will consider $\mathbb{R}^3$ with coordinates $x^i, i = 1,2,3$ with the standard metric $g_{ij} = \delta_{ij}$. We will write $A = A_i dx^i \in \Lambda^1$, in particular $dA = d(A_i dx^i) = \partial_j A_i dx^j \wedge dx^i$. The Hodge operator is given by $*dx^i = \varepsilon_{ijk}\, dx^j \wedge dx^k$ and $*(dx^j \wedge dx^k) = \varepsilon_{jki}\, dx^i$, where ε_{ijk} is totally antisymmetric tensor determined by $\varepsilon_{123} = 1$.

Define a metric on the space of U(1)-connections $\mathcal{C} := \mathcal{C}(M)$ by setting $(A,B) = \int_M A \wedge *B$ for any pair of 1-forms $A, B \in \Lambda^1(M)$. We can rewrite the action as $S'_{CS}(A) = (A, DA)$, where $D = *d$ is the curl operator on 1-forms. The partition function is equal to

$$Z = \int_{\mathcal{C}(M)} e^{ik'(A,DA)} \mathcal{D}A,$$

where $k' = \frac{k}{4\pi}$ for an integer k. The Wilson correlator on a collection of loops $\gamma_a\colon S^1 \to S^3$ corresponding to the characters $n_a\colon \mathrm{U}(1) \to \mathrm{U}(1)$ is equal to

$$\langle \prod_a \exp\left(in_a \int_{\gamma_a} A\right)\rangle = \frac{1}{Z}\int_{\mathcal{C}} \exp\left(ik'(A,DA) + \sum_a in_a \int_{\gamma_a} A\right)\mathcal{D}A,$$

where by definition $\int_{\gamma_a} A := \int_{S^1} \gamma_a^* A = \int_{S^1} A_i \dot\gamma_a^i dt$.

Note that $\ker D = \ker d$ is equal to the set of closed 1-forms on S^3. Let $d^*\colon \Lambda^i \to \Lambda^{i-1}$ be the formal adjoint to the de Rham operator d, it is defined by by the equality $(dA, B) = (A, d^*B)$. One has a formula $d^* = \pm * d *$. Using Hodge theory, it is one formally checks one has $\ker d = (\operatorname{im} d^*)^\perp$ and dualy $\ker d^* = (\operatorname{im} d)^\perp$, where the orthogonal complements are with respect to the inner product $(-,-)$. Hence one can rewrite the orthogonal decomposition as

$$\Lambda^1(M) = \ker D \oplus \operatorname{im}(d*),$$

so the restriction $D\colon \operatorname{im}(d*) \to \operatorname{im}(d*)$ is invertible.

Note that $\operatorname{im} d \subset \Lambda^1_M$ corresponds to the infinitisimal gauge action: given $\delta g \in \Lambda^0_M$ one has $\delta A = d(\delta g)$. Since $H^1(S^3;\mathbb{R}) = 0$, zero modes of D coincide with infinitisimal gauge action. By the same argument $\operatorname{im} D = \operatorname{im}(d^*)$ is equal to $\ker d^* \subset \Lambda^1(M)$. The choice of orthogonal $\ker(d*)$ to $\ker D \subset \Lambda^1(M)$ corresponds to well-known gauge fixing $d * A = 0$ equivalent to $\sum \partial_{x^i} A_i = 0$.

The Green function of the curl operator on $\mathbb{R}^3$ is well-known. For $x, y \in \mathbb{R}^3$, set

$$G_{ij}(x,y) = \varepsilon_{ijk} \frac{1}{4\pi} \frac{x_k - y_k}{|x-y|^3}.$$

Given $f_j(x) dx^j \in \ker(d*)$, let

$$g_i(x) = \int_{\mathbb{R}^3} G_{ij}(x,y) f_j(y) d^3 y.$$

Then a straightforward computation shows that

$$D(g_i(x)dx^i) = f_i(x) dx^i.$$

Let us redefine the path integral by integration only over the fields $A \in \ker(d*)$. As a corollary of (6.0.3) we obtain:

$$\langle A_i(x) A_j(y) \rangle = \frac{i}{2k} \varepsilon_{ijk} \frac{x_k - y_k}{|x-y|^3}.$$

Since the Wilson correlator is given by inserting the source term in the action, by (6.0.2) we obtain

$$\langle \prod_a \exp\left(i n_a \int_{\gamma_a} A \right) \rangle = \exp\left(\frac{i}{2k} \sum_{a,b} n_a n_b \int_{t_a, t_b \in S^1} \varepsilon_{ijk} \frac{\gamma_a^k(t_a) - \gamma_b^k(t_b)}{|\gamma_a(t_a) - \gamma_b(t_b)|^3} \dot\gamma_a^i \dot\gamma_b^j dt_a dt_b \right).$$

On the other hand one can check that

$$\Phi(\gamma_a, \gamma_b) = \frac{1}{4\pi} \int_{t_a, t_b} \varepsilon_{ijk} \frac{\gamma_a^k(t_a) - \gamma_b^k(t_b)}{|\gamma_a(t_a) - \gamma_b(t_b)|^3} \dot\gamma_a^i \dot\gamma_b^j dt_a dt_b = \int_{S^1 \times S^1} f^*(w_2),$$

where $f \colon S^1 \times S^1 \to S^2$ is the map $f(t_a, t_b) = \frac{\gamma_a(t_a) - \gamma_b(t_b)}{|\gamma_a(t_a) - \gamma_b(t_b)|}$ and w_2 is the volume form on S^2 rescaled such that $\int_{S^2} w_2 = 1$. Gauss integral formula says that $\Phi(\gamma_a, \gamma_b)$ is the linking number of knots γ_a and γ_b. Hence we obtain

$$\langle \prod_a \exp\left(i n_a \int_{\gamma_a} A \right) \rangle = \exp\left(\frac{2\pi i}{k} \sum_{a,b} \Phi(\gamma_a, \gamma_b) \right).$$

It follows that the Wilson correlators of abelian Chern-Simons on S^3 are equivalent to linking numbers modulo $k \in \mathbb{Z}$.

Strickly speaking the integrals $\Phi(\gamma_a, \gamma_b)$ might diverge. Though the Willson observables classically make sense for any maps $\gamma_a \colon S^1 \to M$, the quantum expectation values force us to assume that $\cup_a \gamma_a$ is a disjoint union of knots, i.e. a link. In this case one still has a short distance divergence when $a = b$. In order to avoid the problem the natural solution is to work with framed links instead. A frame of a knot is the trivialization of its normal bundle. Equivalently it is an infinitesimal deformation of the knot with no fixed points. We will denote a framed knot by $(\gamma, \delta\gamma)$, so $\gamma + \delta\gamma$ is a knot infinitesimaly close to γ such that the intersection number $\Phi(\gamma, \gamma + \delta\gamma)$ make sense. Given a framed knot $(\gamma, \delta\gamma)$ we define its self-intersection number by $\Phi(\gamma, \gamma + \delta\gamma)$.

8 Projectively flat connections

Fix a manifold X and a complex vector bundle E over X of rank N.

Definition 8.0.1. A connection ∇ on the complex vector bundle E is *projectively flat* if it has scalar curvature: $F(\nabla) = \omega \cdot \mathrm{id}_E$, where $\omega \in \Lambda^2_M$.

Recall that there 1-1 correspondence between representations $\pi_1(X, x) \to \mathrm{GL}_N(\mathbb{C})$ and flat connections. We would like to have a similar characterization of projectively flat connections. Clearly a projectively flat connection ∇ gives a flat connection on $\mathbb{P}(E)$ and the monodromy naturally gives a representation $\pi_1(X, x) \to \mathrm{PGL}_n(\mathbb{C})$ for some basepoint $x \in X$.

Exercise Suppose $\det(E)$ admits an N-th root bundle L, i.e. an isomorphism $L^{\otimes N} \simeq \det(E)$ is fixed. Then ∇ induces a natural connection on L compatible with ∇ (Exercise). Check that the induced connection on $E \otimes L^{-1}$ is flat.

Let $p\colon X' \to X$ denotes the $\mathbb{C}^\times$-bundle $T = \mathrm{Tot}(\det(E) - \{0\})$. The bundle $p^*\det(E)$ is provided with natural trivialization $s \in \Gamma(p^*\det(E))$ (namely $s_v = v$ for a vector $v \in \det(E)_x$ over x). Denote by L' the trivial bundle over X' equal to the "preferred" N-th root of $p^*(\det(E))$ over X'. By the above $p^*E \otimes L'^{-1} \simeq p^*E$ is provided with a natural *flat* connection.

To describe ∇' explicitely let $p^*(\nabla)(s) = \mu \cdot s$, then $p^*(F(\nabla))(s) = d\mu$ is equal to $N \cdot \omega$ and hence

$$\nabla' = p^*(\nabla) - \frac{\mu}{N} \cdot \mathrm{id}_N, \tag{8.0.2}$$

is flat and in fact coincides with the description above. One checks that $\nabla'(s) = 0$. We proved the following.

Proposition 8.0.3. *For a projectively flat connection* (E, ∇) *there is a natural construction of a flat* $\mathrm{SL}_N(\mathbb{C})$*-bundle on* $T = \mathrm{Tot}(\det(E) - \{0\})$.

From the fiber sequence $S^1 \to T \to X$ we get an exact sequence

$$\pi_2(X) \to \pi_1(S^1) = \mathbb{Z} \to \pi_1(T) \to \pi_1(X) \to 1.$$

The homomorphism

$$\langle c_1(E), - \rangle\colon \pi_2(X) \to \mathbb{Z} = \pi_1(S^1),$$

evaluated on the spheroid $s\colon S^2 \to X$ is equal to the degree of $s^*(L)$.

(one can prove that the homomorphism is trivial iff L is a pullback under $X \to B\pi_1$.)

Denote its cokernel by $A = \mathbb{Z}/\langle c_1(L), \pi_2(X)\rangle$. In some interesting examples where $c_1(E) = 0$ or $\pi_2(X) = 0$ we have $A = \mathbb{Z}$. Hence any projectively flat connection (E, ∇) gives a central extension

$$0 \to A \to \tilde{\pi}_1 \to \pi_1 \to 1.$$

Recall that central extensions of a group π_1 by A are classified by the group cohomology $H^2(\pi_1; A) \simeq H^2(B\pi_1; A)$. One can describe this central extension geometrically as follows.

9 Line bundles on abelian varieties

Let $V \simeq \mathbb{C}^n$ be a vector space with a lattice $\Gamma \subset V$ of rank $2n$. The quotient $A = V/\Gamma$ is a compact complex torus. The group of holomorphic line bundle over A is denoted by $\mathrm{Pic}(A)$.

1. By the definition of line bundle we have $\mathrm{Pic}(A) = H^1(A;\mathcal{O}^*)$. The exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{e^{2\pi i -}} \mathcal{O}^* \to 1,$$

induces a long exact sequence of cohomology:

$$0 \to H^1(A;\mathbb{Z}) \to H^1(A;\mathcal{O}) \to H^1(A;\mathcal{O}^*) \xrightarrow{c_1} H^2(A;\mathbb{Z}) \to H^2(A;\mathcal{O}) \tag{9.0.1}$$

By the definition $\mathrm{Pic}^0(A) = \ker c_1$. Recall that

$$H^2(A;\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda^2\Gamma,\mathbb{Z}),$$

so any $w \in H^2(A;\mathbb{Z})$ can be extended to a 2-form on $\Gamma \otimes \mathbb{R} = V$. We call w to be of type $(1,1)$ if $w(ix,iy) = w(x,y)$ for all $x, y \in V$. Any $(1,1)$-form w defines a hermitian form h on V: $h(x,y) = w(ix,y) + iw(x,y)$. This is a 1-1 correspondence. The image c_1 is equal to the kernel of the projection $H^2(A;\mathbb{Z}) \hookrightarrow H^2(A;\mathbb{C}) \to H^{0,2}(A) = H^2(A;\mathcal{O})$, it coincides with integral 2-forms of type $(1,1)$, which is loosely denoted by $H^{1,1}(A;\mathbb{Z})$. Note that since any constant (p,q)-form on V gives a closed (p,q)-form on A, the Hodge decomposition $H^n(A;\mathbb{C}) \simeq \oplus_{p+q=n} H^{p,q}(A)$ available even without Kähler structure on A. For $w \in H^{1,1}(A;\mathbb{Z})$ we set $\mathrm{Pic}^w(A)$ to be the set of line bundles L over A with $c_1(L) = w$. Clearly $\mathrm{Pic}^w(A)$ is a torsor over $\mathrm{Pic}^0(A)$.

From (9.0.1) it follows that $\mathrm{Pic}^0(A) \simeq H^1(A;\mathcal{O})/H^1(A;\mathbb{Z})$ is itself a complex torus, equal to the quotient of n-dimensional vector space $H^1(A;\mathcal{O})$ by the lattice $H^1(A;\mathbb{Z})$. We have $\dim A = \dim \mathrm{Pic}^0(A)$ and as we will see $\mathrm{Pic}^0(A)$ should be thought as a dual to A. The isomorphism $\overline{V}^\vee/\Gamma^\vee \to \mathrm{Pic}^0(A)$ can be described explicitely as follows. The natural map

$$H^1(A;\mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R}) \to H^1(A;\mathcal{O}) = \mathrm{Hom}_{\mathbb{C}}(\bar{V},\mathbb{C})$$

sends $\mathrm{Re}\,\bar{l}$ to $\bar{l}$. The exponential sequence $0 \to \mathbb{Z} \to \mathbb{R} \xrightarrow{e^{2\pi i -}} \mathrm{U}(1) \to 1$ induces a map $H^1(A;\mathbb{R}) \to H^1(A;\mathrm{U}(1))$ and sends $\mathrm{Re}\,\bar{l}$ to $e^{2\pi i\,\mathrm{Re}\,\bar{l}}$. By the naturality of long exact sequences it follows that the image of $\bar{l} \in H^1(A;\mathcal{O})$ in $\mathrm{Pic}^0(A) = H^1(A;\mathcal{O}^*)$ is holomorphic bundle corresponding to the unique $\mathrm{U}(1)$-bundle on $A = V/\Gamma$ with monodromy along γ equal to $e^{2\pi i\,\mathrm{Re}\,\bar{l}(\gamma)}$.

2. Equivalently any $L \in \mathrm{Pic}(A)$ can be describe by a collection of meromorphic functions $e_\gamma\colon V \to \mathbb{C}^\times$ for each $\gamma \in \Gamma$ satisfying the cocycle condition:

$$e_{\gamma_1+\gamma_2}(x) = e_{\gamma_1}(\gamma_2.x)e_{\gamma_2}(x),$$

where we use the multiplicative notation for the action of Γ on V. Given the cocycle $e_\gamma(x)$ we define the corresponding line bundle on A as $V \times \mathbb{C}/(x,t) \sim (\gamma.x, e_\gamma(x)t)$. Two line bundles corresponding to cocycles e_γ and e'_γ are isomorphic iff the cocycles differ by a coboundary:

$$e'_\gamma(x) = e_\gamma(x) \cdot \frac{f(\gamma.x)}{f(x)},$$

for some $f \in \mathcal{O}^*(V)$. In other words there is an isomorphism with the group cohomology $\mathrm{Pic}(A) \simeq H^1(\Gamma; \mathcal{O}^*(V))$. Using the exponential short exect sequence again we get an isomorphism of long exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(\Gamma;\mathbb{Z}) & \longrightarrow & H^1(\Gamma;\mathcal{O}(V)) & \xrightarrow{e^{2\pi i -}} & H^1(\Gamma;\mathcal{O}^*(V)) & \xrightarrow{c_1} & H^2(\Gamma;\mathbb{Z}) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1(A;\mathbb{Z}) & \longrightarrow & H^1(A;\mathcal{O}) & \xrightarrow{e^{2\pi i -}} & H^1(A;\mathcal{O}^*) & \xrightarrow{c_1} & H^2(A;\mathbb{Z}) \end{array}$$

3. We are going to describe explicit cocycles which parametrize $\mathrm{Pic}^w(A)$ for the given $w \in H^{1,1}(A;\mathbb{Z})$. Assume α is a map $\Gamma \to \mathrm{U}(1)$ is such that

$$\frac{\alpha(\gamma_1+\gamma_2)}{\alpha(\gamma_1)\alpha(\gamma_2)} = e^{i\pi w(\gamma_1,\gamma_2)}.$$

The right hand side takes value ± 1 and the set of such α is a torsor over all homomorphisms $\mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathrm{U}(1))$. We will show that the pairs (α, w) parametrize $\mathrm{Pic}^w(A)$.

Let $h(x,y) = w(ix,y) + iw(x,y)$ be the hermitian form corresponding to w. Set

$$e_\gamma(x) = \alpha(\gamma)e^{\frac{\pi}{2}h(\gamma,\gamma)+\pi h(x,\gamma)}. \tag{9.0.2}$$

Proposition 9.0.3. *The collection $e_\gamma(x)$ is a cocycle. The line bundle $L_{(\alpha,w)}$ corresponding to the cocycle satisfies $c_1(L) = w$.*

Proof. DIY. □

Note that the set $\{(\alpha, w)\}$ becomes a group if we set $(\alpha, w).(\alpha', w') = (\alpha\alpha', w + w')$.

Theorem 9.0.4 (Appel-Humbert). *The map $(\alpha, w) \rightsquigarrow L_{(\alpha,w)} \in \mathrm{Pic}^w(A)$ induces an isomorphism $\{(\alpha, w)\} \to \mathrm{Pic}(A)$.*

In particular, for $w = 0$ we obtain that there is an isomorphism

$$\mathrm{Hom}(\Gamma, \mathrm{U}(1)) \to \mathrm{Pic}^0(A).$$

In other words, the construction which sends $U(1)$-flat bundle over A to the corresponding holomorphic line bundle is an isomorphism onto $\mathrm{Pic}^0(A)$.

Exercise Prove directly that any $L \in \mathrm{Pic}^0(A)$ admits a unique hermitian metric such that the corresponding Chern connection is flat. It is called hermitian Yang-Mills metric.

This is equivalent to Narasimhan-Seshadri theorem in rank one case. Even in this case it is only one-half of the story: Simpson's correspondence says that there is a bijection between $\mathrm{Hom}(\Gamma, \mathbb{C}^\times)$ and rank one Higgs fields on A.

10 Correspondences and Poincaré line bundle

Assume $A = V/\Gamma$ and $A' = V'/\Gamma'$ are complex tori.

Definition 10.0.1. A correspondence $P \in \mathrm{Cor}(A, A')$ is a line bundle over $A \times A'$ provided with trivializations over $A \times e'$ and $e \times A'$.

Correspondences form a group under tensor product of line bundles.

Definition 10.0.2. A linear map $f\colon V \times V' \to \mathbb{R}$ is *hermitian* if $f(ix, iy) = f(x, y)$.

1. We have the following simple description of correspondences.

Proposition 10.0.3. *There is a bijection between correspondences $P \in \mathrm{Cor}(A, A')$ and hermitian maps $\Gamma \times \Gamma' \to \mathbb{Z}$.*

Proof. By Appel-Humbert theorem P is given by a form w on $\Gamma \times \Gamma'$ of type $(1,1)$ and a map $\alpha\colon \Gamma \times \Gamma' \to \mathrm{U}(1)$. Let $f(a, a') = w((a, 0), (0, a'))$, this defines a hermitian map $f\colon \Gamma \times \Gamma' \to \mathbb{Z}$.

Since P restricts trivially to $e \times A'$ and $A \times e'$ we obtain that $V \times 0$ and $0 \times V'$ are Lagrangian, hence

$$w((a, a'), (b, b')) = f(a, b') - f(b, a').$$

Similarly $\alpha((\gamma, 0)) = 1 = \alpha((0, \gamma'))$, hence

$$\alpha((\gamma, \gamma')) = \alpha((\gamma, 0) + (0, \gamma')) = e^{i\pi f(\gamma, \gamma')}.$$

Clearly the construction is reversible. □

Remark 10.0.4. More precisely $\mathrm{Cor}(A, A')$ is a discrete groupoid equivalent to the set of hermitian maps as above.

Following the notation of the preceding proof, given such f we describe the associated cocycle explicitely. Immediately we have

$$\alpha((\gamma, \gamma')) = e^{i\pi f(\gamma, \gamma')}. \tag{10.0.5}$$

Note that if h is the hermitian form corresponding to w, then

$$h((a, a'), (b, b')) = h(a, b') + \overline{h(b, a')},$$

where $h(a, b') := h((a, 0), (0, b')) = f(ia, b') + if(a, b')$. Then

$$\frac{\pi}{2} h((\gamma, \gamma'), (\gamma, \gamma')) = \pi \operatorname{Re} h(\gamma, \gamma') = \pi f(i\gamma, \gamma'). \tag{10.0.6}$$

Similarly

$$h((a, a'), (\gamma, \gamma')) = h(a, \gamma') + \overline{h(\gamma, a')} = f(ia, \gamma') + if(a, \gamma') + f(i\gamma, a') - if(\gamma, a'). \tag{10.0.7}$$

2. Recall that $H^1(A; \mathbb{Z}) \simeq \Gamma^\vee$ coincides with the space of real 1-forms on V integral over Γ. On the other hand $H^1(A; \mathcal{O})$ is naturally isomorphic to $\bar{V}^\vee$: the space of $(0,1)$-forms on V. The lattice $\Gamma^\vee \subset \bar{V}^\vee$ is given by $(0,1)$-forms with real part integral on Γ. We have a natural isomorphism of complex tori

$$\mathrm{Pic}^0(A) \simeq \overline{V}^\vee / \Gamma^\vee.$$

Define a hermitian map $f\colon V \times \overline{V}^\vee \to \mathbb{R}$ by setting $f(a, \bar{l}) = -\operatorname{Re} \bar{l}(a)$, where $\bar{l}(a) = \overline{l(a)}$ for any $l \in V^\vee$. By definition

$$\bar{l} \in \Gamma^\vee \iff \operatorname{Re} \bar{l}(a) \in \mathbb{Z},$$

for all $a \in \Gamma$. Clearly $f(ia, i\bar{l}) = -f(ia, \overline{i\bar{l}}) = f(a, \bar{l})$. Hence f is hermitian and integral: $f\colon \Gamma \times \Gamma^\vee \to \mathbb{Z}$. By the previous this defines a correspondence

$$\mathcal{P} \in \mathrm{Pic}(A \times \mathrm{Pic}^0(A))$$

called *Poincaré line bundle.*

The bundle $\mathcal{P}$ is universal in the following sense. For each $\bar{l} \in \overline{V}^\vee/\Gamma^\vee$ the restriction $\mathcal{P}|_{A\times\bar{l}} \in \mathrm{Pic}(V/\Gamma)$ is isomorphic to the line bundle $\bar{l} \in \overline{V}^\vee/\Gamma^\vee \simeq \mathrm{Pic}^0(V/\Gamma)$ described above.

To see this one has to recover the cocycle of $\mathcal{P}_{A\times\bar{l}}$ by using the preceding Proposition. Namely, let $\gamma \in \Gamma, \bar{\mu} \in \Gamma^\vee$ and $a \in V, \bar{l} \in \overline{V}^\vee$. Using the above formulas one can check that the cocyle of $\mathcal{P}$ is given by the formula

$$e_{(\gamma,\bar{\mu})}(a, \bar{l}) = \exp i\pi(\mu(\gamma.a) + \bar{l}(\gamma)).$$

The restriction $\mathcal{P}|_{A\times\bar{l}}$ corresponds to the cocycle

$$e'_\gamma(a) = e^{i\pi\bar{l}(\gamma)},$$

obtained by setting $\mu = 0$ in the above formula. It is convenient to change the cocycle e'_γ by the coboundary of the holomorphic function $a \to e^{i\pi l(a)}$ to get an equivalent cocycle e_γ with values in U(1):

$$e_\gamma(a) = e^{2\pi i \,\mathrm{Re}\,\bar{l}(\gamma)}.$$

This computation shows that $\mathcal{P}_{A\times\bar{l}}$ is U(1)-bundle on A with monodromy along the loop γ equal to $e_\gamma = e^{2\pi i\,\mathrm{Re}\,\bar{l}(\gamma)}$.

Now, if $P \in \mathrm{Cor}(A, A')$ is a correspondence, then it induces a morphism $u\colon A' \to \mathrm{Pic}^0(A)$ by mapping $x' \in A'$ to $P_{A\times x'} \in \mathrm{Pic}^0(A)$. As we saw above $\mathrm{Pic}^0(A)$ and Poincaré line bundle $\mathcal{P} \in \mathrm{Cor}(A, \mathrm{Pic}^0(A))$ is a universal pair among such: there is a unique $u\colon A' \to \mathrm{Pic}^0(A)$ such that the correspondences $u^*\mathcal{P}$ and P are canonically isomorphic. To prove it note that $u^*\mathcal{P} \otimes P^{-1} \in \mathrm{Cor}(A, A')$ has trivial restrictions to all slices $A \times x'$.

Definition 10.0.8. If A is a complex tori, then a dual complex tori is a pair of a complex torus $A^\vee$ and a correspondence $P \in \mathrm{Cor}(A, A^\vee)$, such that the induced morphism $A^\vee \to \mathrm{Pic}^0(A)$ is an isomorphism.

We saw that $A^\vee \to \mathrm{Pic}^0(A)$ is an isomorphism $\iff$ the hermitian map $\Gamma \times \Gamma^\vee\mathbb{Z}$ underlying P is a perfect pairing. Hence the notion of duality is symmetric: the swaping coordinates allows to write $P \in \mathrm{Cor}(A^\vee, A)$, thus $A \simeq (A^\vee)^\vee$. So we can refer to $\mathrm{Pic}^0(A)$ and the Poincare line bundle is *the* dual to A. It follows that there is a natural isomorphism $A \simeq \mathrm{Pic}^0(\mathrm{Pic}^0(A))$.

Note also that any morphism $f\colon A' \to A^\vee$ induces the dual $f^\vee\colon A \to A'^\vee$, such that $(f^\vee)^\vee = f$. The maps $A'^\vee \times A' \xleftarrow{(f^\vee,\mathrm{id})} A \times A' \xrightarrow{(\mathrm{id},f)} A \times A^\vee$ are consistent in the sense that $(\mathrm{id}, f)^*\mathcal{P}$ and $(f^\vee, \mathrm{id})^*\mathcal{P}'$ are naturally isomorphic.

11 Some algebraic-geometry

Recall that abelian varieties are sufficiently rigid objects and any morphism preserving the neutral element is a homomorphism. More precisely we have the following theorem.

Theorem 11.0.1. *Any morphism of $A \to B$ of abelian varieties considered as varieties is the composition of a shift in B and a homomorphism.*

Proposition 11.0.2. *Assume the morphism $p\colon E \to S$ is smooth projective and $L \in \mathrm{Pic}(E)$ is such that $L|_{p^{-1}(s)}$ is trivial for any $s \in S$. Then the natural morphism $p^*p_*L \simeq L$ is an isomorphism. In other words L is a pullback of the* line bundle $p_*L \in \mathrm{Pic}(B)$.

Theorem 11.0.3 (Square theorem)**.**

Theorem 11.0.4 (Cube theorem)**.** *Assume $X \times Y \times Z$ is a product of connected projective varieties with point $x_0 \times y_0 \times z_0$ and $L \in \mathrm{Pic}(X \times Y \times Z)$ is such that the restrictions $L|_{X\times Y\times z_0}, L|_{X\times y_0\times Z}, L|_{x_0\times Y\times Z}$ are trivial. Then L is trivial.*

Remark 11.0.5. In a certain sense the theorem says that Pic is a polynomial functor of degree ≤ 2.

As a corollary assume A is an abelian varietiy and $L \in \mathrm{Pic}(A)$. Let $M \in \mathrm{Pic}(A \times A \times A)$ be defined on the fibers as follows

$$M_{x,y,z} = L_{x+y+z} \otimes L^{-1}_{x+y} \otimes L^{-1}_{y+z} \otimes L^{-1}_{x+z} \otimes L_x \otimes L_y \otimes L_z.$$

Clearly $M_{e,*,*}, M_{*,e,*}, M_{*,*,e}$ are trivial bundles, hence M is trivial.

We have an immediate corollary.

Corollary 11.0.6. *Assume $f, g, h\colon M \to A$ are morphisms from a variety M to the abelian vareity A. A map $\mathrm{Pic}(A) \to \mathrm{Pic}(M)$ given by*

$$(f^* + g^* + h^*) - (f + g)^* - (g + h)^* - (f + h)^* + f^* + g^* + h^*$$

is zero.

Let $[n]\colon A \to A$ denotes the multiplication by $n \in \mathbb{Z}$. If $n \neq 0$, then $[n]$ is an isogeny of degree $|n|^{2\dim A}$: $(\mathbb{Z}/n)^{2\dim A} \simeq \ker[n] \subset A$.

Corollary 11.0.7. *For any $L \in \mathrm{Pic}^w(A)$ we have*

$$[n]^*L \simeq L^{\otimes\binom{n+1}{2}} \otimes [-1]^*L^{\otimes\binom{n-1}{2}} \in \mathrm{Pic}^{n^2\cdot w}(A),$$

moreover if $L \in \mathrm{Pic}^0(A)$, then

$$[n]^*L \simeq L^{\otimes n}.$$

Proof. 1. For the first assertion use the previous corollary to $f = [n + 1], g = [1], h = [-1]$ and solve the reccurent relation.

2. Note that if $f\colon A \to A'$ is morphism of complex tori and $f^*\colon \mathrm{Pic}^0(A') \to \mathrm{Pic}^0(A)$ is the induced map, then its differential at identity $df^*\colon H^1(A'; \mathcal{O}) \to H^1(A; \mathcal{O})$ is equal to the induced map of cohomology. It follows that the map $[n]\colon A \to A$ induces the differential $d[n]^*$ equal to the multiplication by n. It remains to note that a map of complex tori is determined by its differential at identity, hence $[n]^*\colon \mathrm{Pic}^0(A) \to \mathrm{Pic}^0(A)$ is the multiplication by n. □

Let $m\colon A \times A \to A$ denotes the group operation. Denote by $m_x\colon A \to A$ the shift by $x \in A$, so $m_x \circ m_y = m_{x+y}$. Let $x \in A$ and $L \in \mathrm{Pic}(A)$, define

$$\Phi(x, L) = m_x^*(L) \otimes L^{-1}.$$

Note that $\Phi(x, -)\colon \mathrm{Pic}(A) \to \mathrm{Pic}(A)$ preserves the trivial line bundle, hence it is a homomorphism. The same holds for $\Phi(-, L)\colon A \to \mathrm{Pic}(A)$. This implies an isomorphism

$$m^*_{x+y}L \simeq m^*_xL \otimes m^*_yL \otimes L^{-1},$$

which follows also from the cube theorem, because it says that for given $x, y \in A$ the bundle

$$L_{x+y+z} \otimes L_{y+z}^{-1} \otimes L_{x+z}^{-1} \otimes L_z$$

in $z \in A$ is trivial. It follows that $\Phi(x, \Phi(y, L))$ is always trivial:

$$\Phi(x, \Phi(y, L)) = \Phi(x, m_y^* L \otimes L^{-1}) = \Phi(x, m_y^* L) \otimes \Phi(x, L)^{-1} = m_{x+y}^* L \otimes m_y^* L^{-1} \otimes m_x^* L^{-1} \otimes L.$$

Let $A = V/\Gamma$.

Proposition 11.0.8. *Assume $L \in \operatorname{Pic}^w(A)$ for some $w \in H^{1,1}(A; \mathbb{Z})$. The image of*

$$\Phi(-, L) \colon A \to \operatorname{Pic}(A)$$

is a closed connected subgroup in $\operatorname{Pic}^0(A)$. The differential

$$d\Phi(-, L) \colon T_e A \to T_e \operatorname{Pic}^0(A),$$

after the identification with a map $V \to \overline{V}^\vee$, is equal to w. In particular, the form w is non-degenerate if and only if $\Phi(-, L)$ is an isogeny onto $\operatorname{Pic}^0(A)$, which has degree $|\det w|$.

Proof. The first assertion is clear. Let $A = V/\Gamma$. Assume $L \in \operatorname{Pic}^w(A)$ is given by the cocycle $e_\gamma(x)$ of the form (9.0.2) for $\gamma \in \Gamma$, $x \in V$ and integral form $w \colon \Gamma \times \Gamma \to \mathbb{Z}$ corresponding to a hermitian form h. Then

$$\frac{e_\gamma(x + a)}{e_\gamma(x)} = e^{\pi h(a, \gamma)}$$

is a cocycle representing $\Phi(a, L) \in H^1(\Gamma; \mathcal{O}^*(V)) \simeq \operatorname{Pic}(A)$. Multiplying by the coboundary $\gamma \rightsquigarrow e^{l(x+\gamma)-l(x)} = e^{l(\gamma)}$ of a holomorphic function, where $l(x) = -\pi \overline{h(a, x)}, x \in V$, we obtain an equivalent cocycle

$$e^{\pi h(a,\gamma) - \pi \overline{h(a,\gamma)}} = e^{i\pi \operatorname{im} h(a,\gamma)} = e^{2\pi i w(a,\gamma)}.$$

Thus the line bundle $\Phi(a, L)$ is given by the cocycle $\gamma \rightsquigarrow e^{2\pi i w(a,\gamma)}$ with constant values in $\mathrm{U}(1)$. By Appel-Humbert theorem this cocycle represents a trivial class iff $a \in \ker w$. In particular $\ker w \subset V$ is the tangent space to the subgroup $\ker \Phi(-, L)$. $\square$

12 Ample line bundles, positivity and polarizations

1. Recall that for a line bundle L over a complex manifold X one attemps to define a natural map $X \to \mathbb{P}(H^0(X; L)^\vee)$ which sends $x \in X$ to the hyperplane of sections vanishing at a. Explicitely, if $e_0, \ldots, e_n$ is a basis of $H^0(X; L)$, then the corresponding map can be described by the formula $x \rightsquigarrow [e_0(x) : \ldots : e_n(x)] \in \mathbb{P}^n$. The map $X \to \mathbb{P}(H^0(X; L)^\vee)$ is not well defined at the *base points* $x \in X$, where all sections vanish.

Definition 12.0.1. The line bundle $L \in \operatorname{Pic}(X)$ is *ample* if the canonical map

$$X \to \mathbb{P}(H^0(X; L^{\otimes k})^\vee)$$

is an embedding for some $k > 0$.

In short L is ample when its power $L^{\otimes k}$ for sufficiently large k admits enough sections two separate all points in X.

Exercise Check that if the map $f\colon X \to \mathbb{P}^n = \mathbb{P}(H^0(X; L)^\vee)$ is well-defined, then one has a natural isomorphism $f^*(\mathcal{O}_{\mathbb{P}^n}(1)) \xrightarrow{\sim} L$. Conversly, if $i\colon X \subset \mathbb{P}^n$, then the canonical embedding corresponding to the bundle $L = i^*\mathcal{O}_{\mathbb{P}^n}(1)$ is naturally isomorphic to $i \iff$ the restriction map $H^0(\mathbb{P}^n; \mathcal{O}(1)) \to H^0(X; L)$ is an isomorphism.

We call a $(1,1)$-form w positive if $w(ix, x) > 0$.

Theorem 12.0.2 (Kodaira). *A line bundle L over compact complex manifold X is ample $\iff$ its Chern class $c_1(L)$ admits a* positive *representative: there is 2-form w of type $(1,1)$ such that $[w] = c_1(L)$ and w is pointwise positive.*

In particular the ampleness of L is the property of $c_1(L)$ and hence $L \otimes M$ is ample for all $M \in \mathrm{Pic}^0(X)$.

2. Usually an abelian variety A is defined as a smooth projective algebraic group, but without specifying the projective embedding. It is well-known that A is always commutative, so it is topologically an torus, hence its is a quotient of $H_1(A; \mathbb{R})/H_1(A; \mathbb{Z})$. To see the complex structure one can consider the isomorphism $A \xrightarrow{\sim} H^0(A; \Omega^1)^\vee / H_1(A; \mathbb{Z})$ given by integration on holomorphic differentials.

Assume $A = V/\Gamma$ is a complex torus. By Appel-Humbert theorem applied to $L \in \mathrm{Pic}^w(A)$, $c_1(L)$ is represented by a $(1,1)$-form w on Γ. Hence Kodaira theorem implies that that L is ample $\iff$ $w \in H^{1,1}(A; \mathbb{Z})$ is positive.

Definition 12.0.3. A polarized abelian variety (A, w) is a complex torus $A = V/\Gamma$ provided with a positive integral form w on Γ.

Remark 12.0.4. In dimension $n \geq 2$ not every complex torus admit a non trivial line bundle L with $c_1(L) \neq 0$, let alone a polarization. General lattice $\Gamma \subset \mathbb{C}^n$ doesn't admit any $(1,1)$-form w, so $H^{1,1}(V/\Gamma; \mathbb{Z}) \simeq \pi_0(\mathrm{Pic}(V/\Gamma))$ can be trivial. In general the *Picard number* $r = \mathrm{rk}\, \pi_0(\mathrm{Pic}(V/\Gamma))$ can take any value in $0 \leq r \leq \binom{n}{2}$.

We call a correspondence $P \in \mathrm{Cor}(A, A)$ *symmetric* if it is unchanged by swaping two factors. Recall that $m\colon A \times A \to A$ is the multiplication, $p_{1,2}\colon A \times A \to A$ are projections and $\Delta\colon A \to A \times A$ is the diagonal map.

Proposition 12.0.5. *Any symmetric correspondence $P \in \mathrm{Cor}(A, A)$ has the form*

$$P = m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1},$$

where $L = \Delta^*P$. If $L \in \mathrm{Pic}^w(A)$, then P corresponds to the hermitian map $w\colon \Gamma \times \Gamma \to \mathbb{Z}$.

Proof. By 10.0.3, a correspondence $P \in \mathrm{Cor}(A, A)$ is a hermitian map

$$w\colon \Gamma \times \Gamma \to \mathbb{Z}.$$

Recall that w defines an integral $(1,1)$-form on $V \times V$ by

$$v((a, a'), (b, b')) = w(a, b') - w(b, a').$$

The correspondence P is symmetric $\iff$ $v((a, a'), (b, b')) = v((b, b'), (a, a'))$. In other words a symmetric correspondence is equivalent to a skew-symmetric hermitian map $w\colon \Gamma \times \Gamma \to \mathbb{Z}$.

Given $L \in \mathrm{Pic}^w(A)$, the correspondence $P' = m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1} \in \mathrm{Pic}(A \times A)$ has the underlying $(1,1)$-form equal to $v = m^*w - p_1^*w - p_2^*w$, where $m\colon V \times V \to V$ is the sum and p_i are the corresponding projections. Hence

$$v((a,a'),(b,b')) = w(a+a',b+b') - w(a,b) - w(a',b') = w(a,b') - w(b,a')$$

and thus the hermitian map of P' is equal to $w\colon \Gamma \times \Gamma \to \mathbb{Z}$.

On the other hand $\Delta^*P' = [2]^*L \otimes L^{-2} = L$. It follows that P and P' are equal. □

Corollary 12.0.6. *A polarization (A,w) is a symmetric correspondence $P \in \mathrm{Cor}(A,A)$ such that $\Delta^*P \in \mathrm{Pic}(A)$ is ample.*

By Proposition 11.0.8, the polarization can be thought as a pair (A,L), where L is an ample line bundle defined up to multiplication by $\mathrm{Pic}^0(A)$. Then it provides an isogeny

$$\Phi(-,L)\colon A \to A^\vee \simeq \mathrm{Pic}^0(A).$$

We call the isogeny *symmetric* if $\Phi(-,L)^\vee\colon A \simeq (A^\vee)^\vee \to A^\vee$ is equal to $\Phi(-,L)$.

Proposition 12.0.7. *A non degenerate form $w \in H^{1,1}(A)$ is equivalent to a symmetric isogeny $A \to A^\vee$ of degree $|\det w|$.*

Proof. 1. Consider a map $m_x\colon A \to A$ given by multiplication by $x \in A$. If A is polarized by an ample $L \in \mathrm{Pic}^w(A)$, then $L \rightsquigarrow m_x^*L \otimes L^{-1}$ gives a map $\Phi(-,L)\colon A \to \mathrm{Pic}^0(A)$. By Proposition 11.0.8, $\Phi(-,L)$ is an isogeny which depends only on w.

2. Given an isogeny $f\colon A \to A^\vee$, the pullback of Poincaré line bundle $\mathcal{P} \in \mathrm{Cor}(A^\vee, A)$ provides a correspondence $f^*\mathcal{P} \in \mathrm{Cor}(A,A)$. If f was symmetric, then so is $f^*\mathcal{P}$. □

Hence a polarization is a symmetric isogeny $A \to A^\vee \simeq \mathrm{Pic}^0(A)$ of the form $x \rightsquigarrow m_x^*L \otimes L^{-1}$ for an ample $L \in \mathrm{Pic}^w(A)$.

13 Principal polarizations and Siegel's upper half-space

Definition 13.0.1. A *principal polarization* is a polarization (A,w), where $|\det w| = 1$.

In other words the induced isogeny $A \to A^\vee$ is an isomorphism equal to $\Phi(-,L)$ for an ample line bundle $L \in \mathrm{Pic}^w(A)$, where $w\colon \Gamma \times \Gamma \to \mathbb{Z}$ is a perfect pairing.

Our main example of principally polarized abelian varieties is given by the Jacobian of a curve. Namely, let Σ be a complex curve of genus g. The lattice $H^1(C;\mathbb{Z}) \subset H^1(\Sigma;\mathcal{O})$ admits a natural $(1,1)$-form given by the cup product followed by the integration over $[\Sigma]$. The cup pairing is perfect. It follows that the complex torus $A = \mathrm{Pic}^0(\Sigma) \simeq H^1(\Sigma;\mathcal{O})/H^1(C;\mathbb{Z})$ is a principally polarized abelian variety.

2. Consider $\mathbb{R}^{2n}$ with the standard sympletic form w and $\mathbb{C}^n$ with the standard complex structure. Start with the following observation. Denote by (V,Γ,ω) a complex vector space V of dimension n with an embedding of lattice $\mathbb{Z}^{2n} \hookrightarrow V$ and $(1,1)$-form ω such that $\omega|_{\mathbb{Z}^{2n}}$ is the standard integral symplectic form. Then the groupoid of such objects $\{(V,\mathbb{Z}^{2n},\omega)\}$ is discrete, i.e. the authomorphism group of $(V,\mathbb{Z}^{2n},\omega)$ is trivial and one wants to under stand the set $\{V,\mathbb{Z}^{2n},\omega\}/\mathrm{iso}$. It enjoys a natural action $\mathrm{Sp}(2n,\mathbb{R})$ such that the subgroup $\mathrm{Sp}(2n,\mathbb{Z})$ acts via the precomposition of the lattice embedding.

The set of isomorphism classes $\{(V,\mathbb{Z}^{2n},\omega)\}/\mathrm{iso}$, admit two equivalent descriptions. The first is given by the set of complex structures I on $\mathbb{R}^{2n}$ such that the standard form w has type

(1, 1). It is a subset of the set of all complex structures $\mathrm{GL}(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C})$. The action by $a \in \mathrm{Sp}(2n,\mathbb{R})$ is given by aIa^{-1}.

The second description is given by the set of lattice embeddings $i\colon \mathbb{Z}^{2n} \hookrightarrow \mathbb{C}^n$ such that (1) the last n basis elements map to the basis of $\mathbb{C}^n$ and (2) the induced 2-form $i_*(w)$ on $\mathbb{C}^n$ has type (1, 1). The first condition reflects the fact that we are interested in lattices up to the left action by $\mathrm{GL}(n,\mathbb{C})$. The lattice embedding can be written as $(Z|\,\mathrm{id}_n)$, where $Z \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ equal to $X + iY$ for some real matrices X, Y with invertible Y. Thus the right action by $\mathrm{Sp}(2n,\mathbb{R})$ is given by the formula

$$(Z|\,\mathrm{id}).\begin{pmatrix} A & B \\ C & D \end{pmatrix} = ((ZB + D)^{-1}(ZA + C)|\,\mathrm{id}).$$

One can use transposition in order to obtain more recognizible left action:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}.(Z^*|\,\mathrm{id}) = (AZ^* + B)(CZ^* + D)^{-1}.$$

In order to agree with the literature it might be better to compare the results in terms of Z^*. Our goal is to express the conditions on the induced form on $\mathbb{C}^n$ in terms of $Z \in \mathrm{Mat}_{n\times n}(\mathbb{C})$.

We can write

$$(Z|\,\mathrm{id}) = \begin{pmatrix} X & \mathrm{id} \\ Y & 0 \end{pmatrix}$$

and its inverse

$$\begin{pmatrix} X & \mathrm{id} \\ Y & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & Y^{-1} \\ \mathrm{id} & -XY^{-1} \end{pmatrix}.$$

Let $X' = Y^{-1}$ and $Y' = -XY^{-1}$. The matrix of the standard 2-form on $\mathbb{R}^{2n}$ is $W = \begin{pmatrix} 0 & -\mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$ and that of the complex structure $I = \begin{pmatrix} 0 & -\mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$. The induced form $i_*(w)$ in the basis given by standard identification $\mathbb{C}^n \simeq \mathbb{R}^n \oplus I\mathbb{R}^n$, has the following matrix

$$\begin{pmatrix} 0 & \mathrm{id} \\ X'^T & Y'^T \end{pmatrix}\begin{pmatrix} 0 & -\mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}\begin{pmatrix} 0 & X' \\ \mathrm{id} & Y' \end{pmatrix},$$

which is equal to

$$W' := \begin{pmatrix} 0 & X' \\ -X'^T & Y'^T X' - X'^T Y' \end{pmatrix}.$$

The condition that W' has type (1, 1) is $-IW'I = W'$. The positivity of W' is equivalent to $-IW' > 0$ (positive definite). One checks that the first condition is equivalent to X', Y' and hence X, Y to be symmetric matrices. The second condition is equivalent to $-Y > 0$. The convex complex manifold of all symmetric complex matrices $Z^* \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ such that the imaginary part of Z^* is positive definite, is called *Siegel's upper halfspace* $\mathcal{H}_n$. We have $\dim \mathcal{H}_n = n(n-1)/2$. Given $Z^* = X - iY \in \mathcal{H}_n$ the form $i_*(W)$ depends only on the imaginery part:

$$W' = \begin{pmatrix} 0 & Y^{-1} \\ -Y^{-1} & 0 \end{pmatrix}.$$

Theorem 13.0.2. *The stack of principally polarized abelian varieties of dimension n is isomorphic to the quotient* $\mathrm{Sp}(2n,\mathbb{Z}) \setminus \mathcal{H}_n$.

Proof. Above we have shown that the moduli of principally polarized abelian varieties A with a chosen standard symplectic basis $\mathbb{Z}^{2n} \simeq H_1(A;\mathbb{Z})$ is isomorphic to the complex manifold $\mathcal{H}_n$. The action by $\mathrm{Sp}(2n,\mathbb{Z})$ simply change the standard symplectic basis and the quotient describes the moduli stack of proncipally polarized abelian varieties. □

TOADD $\mathrm{Sp}(2n;R)/\mathrm{U}(n) = \mathcal{H}_n$.

14 Deformations of complex structures

A complex manifold M can be described in two ways. The first consists of a pair (M, I), where $I\colon TM \to TM$ is an integrable almost complex structure, such that $I^2 = -1$ and the commutator of vector fields satisfies

$$[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M.$$

The latter is equivalent to the condition that the $(0,1)$ component of de Rham differential, denoted by d'', squares to zero: $d''^2\colon \Lambda^0(M) \to \Lambda^{0,2}(M)$ vanish. The equation $d''f = 0$ defines the sheaf of holomorphic functions and makes M a smooth manifold with a a sheaf of rings over $\mathbb{C}$. The integrability of I is equivalent to the statement that locally M, as a ringed manifold, is isomorphic to a ball in $(\mathbb{C}^n, I)$ equiped with the sheaf of holomorphic functions.

1. Recall that a complex structure on a real vector space (T, I) is equivalent to the decomposition $T\otimes\mathbb{C} = T' \oplus T''$ such that $\overline{T'} = T''$. In our notation $T' = T^{1,0}$ and $T'' = T^{0,1}$. The complex structure I_t for some parameter $t \in B$ such that $I_0 = I$, is a decomposition

$$T\otimes\mathbb{C} = T'_t \oplus T''_t,\ \overline{T'_t} = T''_t.$$

For small t, T'_t is a graph of a map $\overline{\mu}\colon T' \to T''$ and the projection $p'_t\colon T'_t \to T'$ is an isomorphism. Thus any vector in T'_t has a form $x' + \overline{\mu}(x')$ for some $x' \in T'$, so we can write $p'^{-1}_t(x') = x' + \overline{\mu}(x')$. Likewise any vector T''_t has a form $x'' + \mu(x'')$ for some $x'' \in T''$, where $\mu\colon T'' \to T'$ is the conjugation of $\bar{\mu}$.

The columns of the matrix $\begin{pmatrix} 1 & \mu \\ \overline{\mu} & 1 \end{pmatrix}$, viewed as a map $T' \oplus T'' \to T' \oplus T''$, span T'_t and T''_t, respectively. The complex structure I_t has a matrix expression

$$I_t = \begin{pmatrix} 1 & \mu \\ \overline{\mu} & 1 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 1 & \mu \\ \overline{\mu} & 1 \end{pmatrix}^{-1}.$$

We have

$$\begin{pmatrix} 1 & \mu \\ \overline{\mu} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -\mu \\ -\overline{\mu} & 1 \end{pmatrix}\begin{pmatrix} (1-\mu\overline{\mu})^{-1} & 0 \\ 0 & (1-\overline{\mu}\mu)^{-1} \end{pmatrix},$$

and

$$I_t = i\begin{pmatrix} 1+\mu\overline{\mu} & -2\mu \\ 2\overline{\mu} & -(1+\overline{\mu}\mu) \end{pmatrix}\begin{pmatrix} (1-\mu\overline{\mu})^{-1} & 0 \\ 0 & (1-\overline{\mu}\mu)^{-1} \end{pmatrix}.$$

Let us note the quadratic expansion:

$$I_t = I + 2i\begin{pmatrix} \mu\overline{\mu} & -\mu \\ \overline{\mu} & -\overline{\mu}\mu \end{pmatrix} + o(t^2). \tag{14.0.1}$$

Dually, the decomposition $\Lambda\otimes\mathbb{C} = \Lambda'_t \oplus \Lambda''_t$ is given by the inverse of

$$\begin{pmatrix} 1 & \mu \\ \overline{\mu} & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & \overline{\mu}^T \\ \mu^T & 1 \end{pmatrix},$$

and is equal to

$$\begin{pmatrix}(1-\overline{\mu}^T\mu^T)^{-1} & 0\\ 0 & (1-\mu^T\overline{\mu}^T)^{-1}\end{pmatrix}\begin{pmatrix}1 & -\overline{\mu}^T\\ -\mu^T & 1\end{pmatrix},$$

where $\mu^T\colon \Lambda' \to \Lambda''$ is the dual to $\mu\colon T'' \to T'$. Note that μ and μ^T represent the same element in $\Lambda^{0,1}\otimes T'$, we distinct them only for convenience.

2. Assume (M, I) is a complex manifold. Denote by T and its dual Λ the tangent and cotangent bundle respectively. The almost complex deformation I_t of I is given by $\mu\colon T'' \to T'$. We will identify μ and μ^T with an element $\tau \in \Lambda^{0,1}\otimes T'$.

The structure I_t is integrable $\iff$ $[T''_t, T''_t] \subset T''_t$. Locally $\tau = d\overline{z}^i \otimes \tau_i$ for some $\tau_i \in T'$ and the integrability is equivalent to the vanishing of

$$[\overline{\partial}_i + \tau_i, \overline{\partial}_j + \tau_j] = \overline{\partial}_i(\tau_j) - \overline{\partial}_j(\tau_i) + [\tau_i, \tau_j].$$

Invariantly the latter is equivalent to vanishing of Frölicher–Nijenhuis tensor

$$d''\tau + \frac{1}{2}[\tau,\tau] = 0 \in \Lambda^{0,2}\otimes T'. \tag{14.0.2}$$

Here the bracket is defined in the usual way: $[\tau, -]$ is the Lie derivative of degree 1. Namely,

$$[d\overline{z}^i\tau_i, -] = d\overline{z}^i[\tau_i, -] = d\overline{z}^i\mathcal{L}_{\tau_i}(-).$$

For instance

$$[\tau,\tau] = [d\overline{z}^i\tau_i, d\overline{z}^j\tau_j] = [\tau_i,\tau_j]d\overline{z}^i\wedge d\overline{z}^j,$$

since $[\tau_i, d\overline{z}^j] = 0$. Hence I_t is integrable $\iff$ τ satisfies the Maurer-Cartan equation (14.0.2).

Given a function f, let

$$u' = p'_t(d'_t f),\ \ v'' = p''_t(d''_t f),$$

or

$$d'_t f = u' - \tau(u'),\ \ d''_t f = v'' - \tau(v'').$$

Since $df = d'f + d''f = d'_t f + d''_t f$, we have

$$\begin{pmatrix}1 & -\overline{\tau}\\ -\tau & 1\end{pmatrix}\begin{pmatrix}u'\\ v''\end{pmatrix} = \begin{pmatrix}d'f\\ d''f\end{pmatrix},$$

or

$$\begin{pmatrix}u'\\ v''\end{pmatrix} = \begin{pmatrix}(1-\overline{\tau}\tau)^{-1} & 0\\ 0 & (1-\tau\overline{\tau})^{-1}\end{pmatrix}\begin{pmatrix}1 & \overline{\tau}\\ \tau & 1\end{pmatrix}\begin{pmatrix}d'f\\ d''f\end{pmatrix}.$$

Thus $v'' = (1-\tau\overline{\tau})^{-1}(d''f + [\tau, f])$. The function f is holomorphic with respect to I_t, if and only if, $d''f + [\tau, f] = 0$. We obtain a commutative diagram

$$\begin{array}{ccccc}\Lambda^0 & & \xrightarrow{\quad\text{id}\quad} & & \Lambda^0\\ {\scriptstyle d''+[\tau,-]}\downarrow & & & & \downarrow{\scriptstyle d''_t}\\ \Lambda^{0,1} & \xrightarrow[(1-\tau\overline{\tau})^{-1}]{} & \Lambda^{0,1} & \xrightarrow[p''^{-1}_t]{} & \Lambda^{0,1}_t\end{array} \tag{14.0.3}$$

The bottom composition naturally extends to an isomorphism of graded algebras over $\Lambda^0(M)$:

$$\Lambda^{0,*}(M) \to \Lambda^{0,*}_t(M).$$

We already saw that τ acts on $\Lambda^{0,*} \otimes T'$ by a Lie derivation $[\tau, -]$ of degree 1. Locally $\tau = d\bar{z}^i \otimes \tau_i$. In particular $[\tau, f] = d\bar{z}^i[\tau_i, f]$ and $[\tau, d\bar{z}^k] = 0$, the Leibniz rule completely determines $\mathcal{L}_\tau$.

Thus we obtain a graded commutative algebra with a derivation of degree 1 over $\Lambda^0(M)$:

$$(\Lambda^{0,*}(M), d'' + [\tau, -]).$$

We can reinterpret the integrability of I_t as a deformation of Dolbeault complex cdga.

Proposition 14.0.4. *The deformation I_t given by $\tau \in \Lambda^{0,1} \otimes T'$ is integrable, if and only if, $d''\tau + \frac{1}{2}[\tau, \tau] = 0$, i.e. $(\Lambda^{0,*}(M), d'' + [\tau, -])$ is a cdga.*

15 Deformation of Kähler polarization

1. Assume (M, I, w) is a Kähler manifold with holomorphic U(1)-bundle L such that the curvature of the Chern connection ∇ on L is equal to iw. We are interested in deformations I_t of I with respect to w, i.e. w is of type $(1,1)$ with respect to I_t. We think of the deformation I_t in temrs of a trivial family of C^∞-manifolds $M \times B$ over some parameter space B, so $t \in B$ is a point. It follows that such I_t naturally induces a holomorphic structure on L: by the definition $\nabla'' = \frac{1}{2}(1 + iI_t)\nabla$ and we have $\nabla''^2 = 0$ because $[\nabla, \nabla] = iw$ is of type $(1,1)$ with respect to I_t.

We will focus on first order deformation of I, i.e. $I_t = I + \delta I + o(t)$. According to (14.0.1) we can write $\delta I = 2i(\overline{\tau} - \tau)$ for some $\tau \in \Lambda'' \otimes T'$ such that $d''\tau = 0$. Then

$$\delta(\overline{\partial}) = \frac{1}{2} i \cdot \delta I \circ d = \tau - \overline{\tau},$$

is a derivative of degree 1. Similarly we have $\delta(\nabla'') = \nabla_\tau - \nabla_{\overline{\tau}}$.

Suppose $s \in L$ is a local holomorphic section, i.e. $\nabla''(s) = 0$. Consider the problem of extension of s to a holomorphic section $s + \delta s$. In other words we are interested in solutions of the equation

$$\delta(\nabla'' s) = \delta(\nabla'')s + \nabla''\delta s = \nabla'_\tau s + \nabla''\delta s = 0 \in \Lambda^{0,1} \otimes L. \tag{15.0.1}$$

The consistency of the equation, i.e. the existence of local solutions, is equivalent to

$$0 = \nabla''(\nabla'_\tau s) = [\nabla'', \nabla'_\tau](s) = -\iota_\tau[\nabla'', \nabla'](s) = i/2 \cdot \iota_\tau w \cdot s \in \Lambda^{0,2} \otimes L. \tag{15.0.2}$$

By our assumption $w(I_t -, I_t -) = w(-,-)$, which infinitesimally is $w(\delta I -, -) + w(-, \delta I -) = 0$ $\iff$ $w(\delta I -, -)$ is a symmetric tensor. Since w is a real form of type $(1,1)$,

$$w(\delta I -, -) = 2i \cdot w(\overline{\tau} -, -) - 2i \cdot w(\tau -, -),$$

is a difference of different types, so the vanishing of the antisymmetric part of $w(\delta I -, -)$ is equivalent to $\iota_\tau w = 0 \in \Lambda^{0,2}$. This implies (15.0.2).

It will be convenient to use the isomorphism $w \colon T' \to \Lambda''$ to identify the symmetric tensor $w(\tau -, -) \in \mathrm{Sym}^2(\Lambda^{0,1})$ with a tensor in $\mathrm{Sym}^2(T')$. Locally

$$w = w_{i\bar{j}} dz^i d\bar{z}^j, \tau = \tau^b_{\bar{a}} \partial_b \otimes d\bar{z}^a,$$

so

$$w(\tau -, -) = w_{i\bar{j}} dz^i (\tau^b_{\bar{a}} \partial_b \otimes d\bar{z}^a) d\bar{z}^j = w_{i\bar{j}} \tau^i_{\bar{a}} d\bar{z}^a \otimes d\bar{z}^j = \tau^a_{\bar{i}} w_{a\bar{j}} d\bar{z}^i \otimes d\bar{z}^j.$$

If

$$w(\tau w^{-1}(-), w^{-1}(-)) = G^{ab} \partial_a \otimes \partial_b,$$

for some symmetric G^{ab}, then

$$w(\tau-,-) = G^{ab} w_{a\bar{i}} w_{b\bar{j}} d\bar{z}^i \otimes d\bar{z}^j,$$

hence for all i,j we have $\tau_{\bar{i}}^a w_{a\bar{j}} = G^{ab} w_{a\bar{i}} w_{b\bar{j}}$, or by lifting indexes by the multiplication on the inverse $w^{k\bar{j}}$, we obtain

$$\tau_{\bar{i}}^k = G^{ak} w_{a\bar{i}}. \tag{15.0.3}$$

2. The map $\nabla'_\tau\colon L \to \Lambda^{0,1} \otimes L$ is a differential operator of degree ≤ 1. Denote the sheaf of holomorphic differential operators of degree $\leq k$ acting on L by $\mathrm{Diff}_{\leq k}(L)$. By the definition $\mathrm{Diff}_{\leq}(L) = \mathcal{O}$ is $\mathcal{O}$-linear operators, while $\phi \in \mathrm{Diff}_{\leq k}(L)$ is determined by the condition $[\phi, f] \in \mathrm{Diff}_{<k}(L)$ for any $f \in \mathcal{O}$. Thus $\mathrm{Diff}(L) = \cup_k \mathrm{Diff}_{\leq k}(L)$ is a sheaf of filtered algebras. We have

$$0 \to \mathrm{Diff}_{<k}(L) \to \mathrm{Diff}_{\leq k}(L) \xrightarrow{\sigma} \mathrm{Sym}^k(T') \to 0,$$

where the projection σ is the principal symbol map. Thus the associated graded to $\mathrm{Diff}(L)$ is the free symmetric algebra $\mathrm{Sym}(T')$ generated by the tangent bundle. Recall that the canonical extension

$$0 \to \mathcal{O} \to \mathrm{Diff}_{\leq 1}(L) \to T' \to 0,$$

is called the Atiyah class of L, it lies in $\mathrm{Ext}^1_M(T, \mathcal{O}) \simeq H^1(M;\Omega^1)$ and is equal to the class of curvature $[F(\nabla)] \in H^{1,1}$. In our case $[F(\nabla)]$ is proportional to $[iw]$.

By (15.0.2), $\nabla''(\nabla'_\tau) = 0$ and we have a well-defined class $[\nabla'_\tau] \in H^1(M; \mathrm{Diff}_{\leq 1}(L))$. We have a long exact sequence

$$0 \to H^0(\mathcal{O}) \to H^0(\mathrm{Diff}_{\leq 1}(L)) \xrightarrow{\sigma} H^0(T) \xrightarrow{F(\nabla)} H^1(\mathcal{O}) \to H^1(\mathrm{Diff}_{\leq 1}) \xrightarrow{\sigma} H^1(T). \tag{15.0.4}$$

Note that $\sigma(\nabla'_\tau) = \tau$, thus $\sigma([\nabla'_\tau])$ is equal (up to an irrelevant multiplier) to the Kodaira-Spencer class in $H^1(M;T)$ of the deformation δI.

Following Hitchin we interpret the condition on $(s, \delta s)$ in (15.0.1) as a cocycle condition in a certain complex. Fix s such that $\nabla'' s = 0$. Consider a complex of $\mathcal{O}$-modules

$$A(s, \mathrm{Diff}_{\leq 1}(L)) := \mathrm{Cone}(\mathrm{Diff}_{\leq 1}(L) \xrightarrow{s} L)[-1],$$

where the shift ensures that L is in degree 1. The complex $A(s, \mathrm{Diff}_{\leq 1}(L))$ naturally admits a Dolbeault resolution denoted by

$$A^*(\mathrm{Diff}_{\leq 1}, s) = \mathrm{Cone}(\mathrm{Diff}_{\leq 1}(L) \xrightarrow{s} L) \otimes \Lambda^{0,*}.$$

Explicitely we have

$$A^p(\mathrm{Diff}_{\leq 1}, s) = \Lambda^{0,p} \otimes \mathrm{Diff}_{\leq 1}(L) \oplus \Lambda^{0,p-1} \otimes L,$$

where the differential $d_s\colon A^p \to A^{p+1}$ is given by the formula

$$d_s(D, w) = (\nabla''(D), \nabla''(w) - (-1)^p D(s)),$$

for all $D \in \Lambda^{0,p} \otimes \mathrm{Diff}_{\leq 1}(L)$ and $w \in \Lambda^{0,p-1} \otimes L$. The (15.0.1) is equivalent to the cocycle condition on

$$(\nabla'_\tau, \delta s) \in A^1(\mathrm{Diff}_{\leq 1}, s).$$

Recall that $M \times B$ is a smooth family of complex structures (M, I_t) such that w is of type $(1,1)$ for all $I_t, t \in B$. The class $w \in H^1(M, I_t; \Omega^1)$ is well defined. Furthermore, we fixed U(1)-bundle L on M with unitary connection ∇ such that $F(\nabla) = iw$ Then for all t the bundle L

admits a natural holomorphic structure with respect to I_t and after passing to sufficiently large power of L, if necessary, one has a complex vector bundle $H^0(M, I_t; L)$ over B. Given tangent vector $\delta t \in T_{t\in B}$ one has the corresponding first order deformation δI of I, i.e.

$$\delta I = 2i(\overline{\tau} - \tau), \ [\tau] = \mathrm{KS}(\delta t) \in H^1(M, I_t; T),$$

where $\mathrm{KS}\colon T_{t\in B} \to H^1(M, I_t; T)$ is the Kodaira-Spencer map. We have the main theorem.

Theorem 15.0.5 (Hitchin)**.** *Suppose that for each $t \in B$, tangent vector $\delta t \in T_{t\in B}$ and $s \in H^0(M, I_t; L)$, one has a class*

$$A(s, t, \delta t) \in \mathbb{H}^1(M; A(\mathrm{Diff}_{\leq 1}(L), s)),$$

smoothly dependend on $(s, t, \delta t)$ and such that $\sigma(A(s, t, \delta t)) = \mathrm{KS}(\delta t) \in H^1(M, I_t; T)$.. Assume the map

$$H^0(M, I_t; T) \xrightarrow{[w]} H^1(M, I_t; \mathcal{O})$$

is an isomorphism, then there is a natural connection on the projective bundle $\mathbb{P}(H^0(M, I_t; L))$.

Proof. We will work at the given point $I = I_t$. By the assumption on w, the long exact sequence (15.0.4) implies that $H^0(\mathcal{O}) \to H^0(\mathrm{Diff}_{\leq 1}(L))$ is an isomorphism and the symbol map $\sigma\colon H^1(\mathrm{Diff}_{\leq 1}(L)) \to H^1(T)$ is an inclusion. The tangent vector δt defines a variation δI equal to $\delta I = 2i(\overline{\tau} - \tau)$ for some d''-closed $\tau \in \Lambda'' \otimes T'$. Assume $(D, w) \in A^1(\mathrm{Diff}_{\leq 1}, s)$ is a representative of $A(s, t, \delta t)$, for some $D \in \Lambda^{0,1} \otimes \mathrm{Diff}_{\leq 1}(L)$ and $w \in \Lambda^{0,0} \otimes L$ such that $\nabla'' D = 0$ and $Ds + \nabla'' w = 0$. Since

$$\sigma([D]) = \mathrm{KS}(\delta t) = [\tau] = \sigma([\nabla_\tau]) \in H^1(M; T),$$

and $H^1(\mathrm{Diff}_{\leq 1}) \hookrightarrow H^1(T)$ is an inclusion, we have

$$D - \nabla'_\tau = \nabla'' \Phi$$

for some $\Phi \in \Lambda^0 \otimes \mathrm{Diff}_{\leq 1}(L)$. Hence

$$0 = Ds + \nabla'' w = \nabla'_\tau s + \nabla''(\Phi s + w),$$

where we used that $\nabla''(\Phi s) = \nabla''(\Phi) s$ since $\nabla'' s = 0$. Thus setting $\delta s := w + \Phi s$ gives a solution to (15.0.1). Since Φ is unique up to an element from $H^0(\mathrm{Diff}_{\leq 1}(L)) \simeq H^0(\mathcal{O})$, i.e. a constant, we conclude that δs is unique up to a multiple of s. Because of this ambiguity one get a connection on $\mathbb{P}(H^0(M, I_t; L))$. ☐

3. The map of short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Diff}_{\leq 1}(L) & \longrightarrow & \mathrm{Diff}_{\leq 2}(L) & \xrightarrow{\sigma} & \mathrm{Sym}^2(T) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} & & \downarrow & & \\ 0 & \longrightarrow & L & \xrightarrow{\mathrm{id}} & L & \longrightarrow & 0 & \longrightarrow & 0 \end{array} \tag{15.0.6}$$

gives a short exact sequence of complexes

$$0 \to \mathrm{Cone}(\mathrm{Diff}_{\leq 1}(L) \xrightarrow{s} L)[-1] \to \mathrm{Cone}(\mathrm{Diff}_{\leq 2} \xrightarrow{s} L)[-1] \to \mathrm{Sym}^2(T) \to 0,$$

and hence, setting $A(\mathrm{Diff}_{\leq k}(L), s) = \mathrm{Cone}(\mathrm{Diff}_{\leq k} \xrightarrow{s} L)[-1]$, by inspecting the induced long exact sequence of hypercohomology, we have a boundary map

$$\Psi\colon H^0(M; \mathrm{Sym}^2(T)) \to \mathbb{H}^1(M; A(\mathrm{Diff}_{\leq 1}, s)).$$

One can compute $\Psi(G)$ for some $G = G^{ij}\partial_i \otimes \partial_j \in H^0(M; \mathrm{Sym}^2(T))$ using zig-zag lemma applied for the corresponding Dolbeault resolutions. Recall that $\Psi(G)$ is represented in $A^1(s, \mathrm{Diff}_{\leq 1}(L))$ by a lift of G to $A^0(s, \mathrm{Diff}_{\leq 2}(L))$ followed by application of d_s. The second order differential operator $\nabla'_i \circ (G^{ij}\nabla'_j)$ has symbol $G = G^{ij}\partial_i \otimes \partial_j$. Then the class of $[\nabla'', \nabla'_i \circ G^{ij}\nabla_j]$ is $\Psi(G)$. Then

$$[\nabla'', \nabla'_i \circ G^{ij}\nabla'_j] = -i/2 \cdot (\iota_{\partial_i} w \cdot G^{ij}\nabla'_j + \nabla'_i(G^{ij}\iota_{\partial_j} w))$$

is a first order differential operator with the symbol equal to

$$-i/2 \cdot \iota_{\partial_i} w \cdot G^{ij}\partial_j = -i/2 \cdot w_{i\bar{a}} G^{ij} d\bar{z}^a \cdot \partial_j.$$

Similarly to 15.0.5, a family of complex structures on M parametrized by $t \in B$, defines a symmetric tensor $G(t, \delta t)$ (15.0.3), smoothly dependend on $t \in B$ and $\delta t \in T_{t\in B}$.

Corollary 15.0.7. *Assume the symmetric tensor G is holomorphic, then*

$$\sigma(\Psi(2iG)) = [\tau] \in H^1(M; T).$$

In particular, if $G = (t, \delta t)$ is holomorphic for each $t \in B$ and $\delta t \in T_{t\in B}$, then

$$\Psi(2iG) \in H^1(A^*(s, \mathrm{Diff}_{\leq 1}(L)))$$

satisfies the conditions of Theorem 15.0.5 and provides $\mathbb{P}(H^0(M, I_t; L))$ with a natural projective connection.