Notes on Chern-Simons theory

Abstract

These notes provide a review of the Chern-Simons topological quantum field theory from the point of view of the Atiyah-Segal axioms. The goal is to make these notes accessible to mathematicians and physicists alike.

1 Introduction

Topological quantum field theories (TQFTs) have far reaching applications not only in the subject of their origin, theoretical physics, but also in mathematics.

In physics, they find applications in condensed matter physics, related to conformal field theories.

In mathematics, TQFTs are a highly refined tool for studying the topology of manifolds.

(Under construction....)

1.1 Preliminaries on oriented manifolds

We will review some elementary facts about oriented manifolds.

(Under construction...)

1.2 Conventions

1. Fix a base field k.

2. All manifolds are smooth and equipped with an orientation, but not necessarily connected.

3. If Y is an oriented manifold, then $\overline{Y}$ has the same underlying manifold as Y equipped with the opposite orientation.

4. A closed manifold is a compact manifold without boundary, denoted Σ.

5. $I = [0, 1]$ denotes the closed unit interval equipped with the standard orientation.

6. Let Y be a manifold and $\Sigma \hookrightarrow Y$ be a closed submanifold of codimension 1 — both equipped with an orientation. At $x \in \Sigma$, let $\{v_1, \ldots, v_{d-1}\}$ be

a positively oriented basis of $T_x\Sigma$. A vector $w \in T_xY$ is called a positive normal if $\{v_1, \ldots, v_{d-1}, w\} \subset T_xY$ is a positively oriented basis.

If $\Sigma \subset \partial Y$ be a connected component, then

a. Σ is an in-boundary if w points inwards.

b. Σ is an out-boundary if w points outwards.

2 Atiyah-Segal axioms

Definition 2.1. Let Σ_0 and Σ_1 be closed, oriented $(d-1)$-manifolds. An oriented cobordism from Σ_0 to Σ_1 is an oriented d-manifold Y together with maps

$$\Sigma_0 \longrightarrow Y \longleftarrow \Sigma_1$$

such that Σ_0 maps diffeomorphically onto the in-boundary of Y, and Σ_1 maps diffeomorphicaly onto the out-boundary of Y. Note that $\partial Y \simeq \overline{\Sigma_0} \coprod \Sigma_1$.

Definition 2.2. Let $f : \Sigma_0 \to \Sigma_1$ be an orientation preserving diffeomorphism. Then cobordism associated to f is

$$\Sigma_1 \xrightarrow{(id,0)} \Sigma_1 \times I \xleftarrow{(f,1)} \Sigma_0$$

We denote this cobordism by I_f.

Given a pair $f : \Sigma_0 \to \Sigma_1$ and $g : \Sigma_1 \to \Sigma_2$ of composable, orientation preserving diffeomorphisms, we obtain a pair of composable cobordisms

$$\Sigma_2 \xrightarrow{(id,0)} \Sigma_0 \times I \xleftarrow{(g,1)} \Sigma_1 \xrightarrow{(id,0)} \Sigma_1 \times I \xleftarrow{(f,1)} \Sigma_0$$

Their composition is defined by

$$\Sigma_2 \xrightarrow{(id,0)} \Sigma_2 \times I \xleftarrow{(gf,1)} \Sigma_0$$

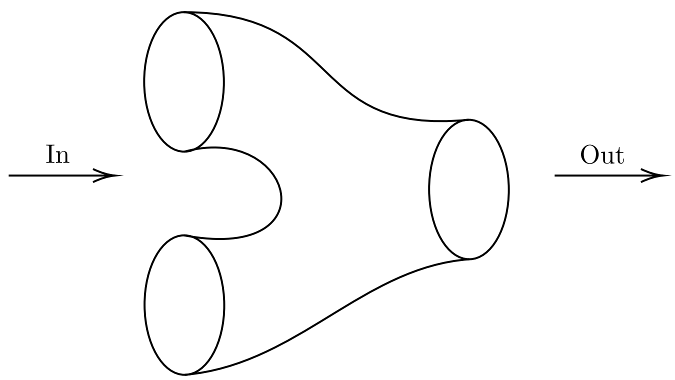
Definition 2.3. We say Y_0 and Y_1 be two oriented cobordisms, from Σ_0 to Σ_1, are equivalent if there exists a diffeomorphism $Y_0 \xrightarrow{f} Y_1$ such that

$$\begin{array}{ccccc} & & Y_0 & & \\ & \nearrow & & \nwarrow & \\ \Sigma_0 & & \downarrow f & & \Sigma_1 \\ & \searrow & & \swarrow & \\ & & Y_1 & & \end{array}$$

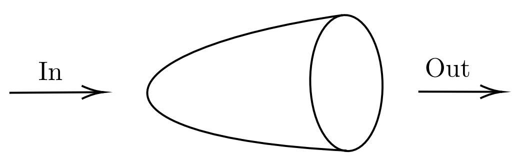
commutes. This defines an equivalence relation on cobordisms between Σ_0 and Σ_1.

Example 2.4. The following are examples of 2-dimensional cobordisms.

1. Pair of pants



2. Cup



3. Cylinder

Each of these cobordisms can be reversed to obtain cobordisms with the in-boundary and the out-boundary exchanged.

Definition 2.5. A topological quantum field theory in dimension d is a "rule" Z that assigns

1. a finite dimensional k-vector space $Z(\Sigma)$ to each closed and oriented $(d-1)$-manifold Σ — *the state space of* Σ.

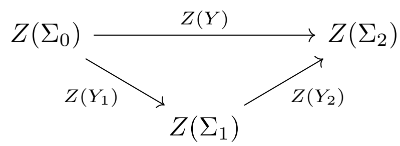
2. a vector $Z(Y) \in Z(\Sigma)$ for each oriented d-manifold Y with boundary Σ — *a state.*

such that

A1. $Z(\overline{\Sigma}) = Z(\Sigma)^\vee$, where $(-)^\vee$ is the k-linear dual.

A2. $Z(\Sigma_0 \coprod \Sigma_1) = Z(\Sigma_0) \otimes_k Z(\Sigma_1)$.

A3. If a d manifold Y is of the form $Y \simeq Y_1 \coprod_\Sigma Y_2$ and $\partial Y \simeq \overline{\Sigma}_0 \coprod \Sigma_1$, then the following diagram of k-vector spaces commutes



A4. Let $\varnothing_{d-1}$ be the empty $(d-1)$-manifold, then $Z(\varnothing_{d-1}) = k$.

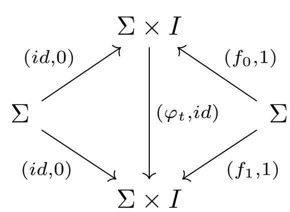
A5. and Σ be an oriented $(d-1)$-manifold, then $Z(\Sigma \times I) = id_{Z(\Sigma)}$.

We note that Z is defined on equivalence classes of cobordisms, not individual cobordisms. The following observations are immediate:

Lemma 2.6. *Every orientation preserving diffeomorphism $f : \Sigma_0 \to \Sigma_1$ induces a k-linear isomorphism $Z(I_f) : Z(\Sigma_0) \to Z(\Sigma_1)$.*

Lemma 2.7. *The group of orientation preserving diffeomorphisms of Σ acts on $Z(\Sigma)$ via its group of components.*

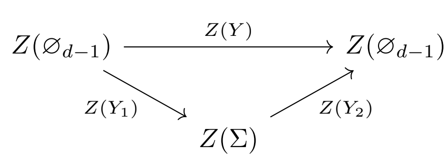
Proof. Let $f_0, f_1 : \Sigma \to \Sigma$ be orientation preserving diffeomorphisms, and $F : \Sigma \times I \to \Sigma$ be a smooth homotopy from f_0 to f_1. This data induces the following commutative diagram



where $\varphi_t \coloneqq f_t f_0^{-1}$. It is evident that the vertical arrow is an orientation preserving diffeomorphism. $\square$

We obtain manifold invariants from Z:

Lemma 2.8. *For a closed d-manifold Y, $Z(Y) \in k$ is a scalar. For every codimension 1 submanifold $\Sigma \hookrightarrow Y$ that decomposes Y into two pieces Y_1 and Y_2, i.e $Y \simeq Y_1 \coprod_\Sigma Y_2$, the following diagram commutes.*



we find that

$$Z(Y) = Z(Y_2) \circ Z(Y_1) = \langle Z(Y_2), Z(Y_1) \rangle$$

where $\langle , \rangle$ *evaluates a dual vector on a vector. In other words,* $Z(Y)$ *is an orientation preserving diffeomorphism invariant of* Y*.*

Lemma 2.9. *Let* f *be an orientation preserving diffeomorphism of* Σ*. Let* $\Sigma_f \coloneqq (\Sigma \times I)/\sim$*, where* $(s, 0) \sim (f(s), 1)$*, for every for* $s \in \Sigma$*. Then*

$$Z(\Sigma_f) = \mathrm{Tr}(Z(I_f))$$

Proof. Exercise. □

Remark 2.10. In short, a d-dimensional TQFT is a symmetric monoidal functor

$$\mathrm{Cob}_d^{\mathrm{or}} \xrightarrow{Z} \mathrm{Vect}_k$$

$\mathrm{Cob}_d^{\mathrm{or}}$ has closed, oriented $(d-1)$-dimensional manifolds as objects and equivalence classes of oriented cobordisms between them as morphisms.

For broader applications, we may equip cobordisms with additional structure, like a Riemannian metric, or a spin structure. In such a case, the theory need not remain topological.

We provide an elementary dictionary between the physical terminology and the mathematical terminology. For a d-dimensional theory Z,

Mathematics	**Physics**
oriented d-manifold Y	spacetime
$Z(Y)$	partition function of Y
oriented $(d-1)$-manifold Σ	space
$Z(\Sigma)$	Hilbert space on Σ
$Z(\Sigma \times I)$	imaginary time evolution, $\exp(-TH)$
$Z(Y)$ independent of the Σ, $Y \simeq Y_1 \coprod_\Sigma Y_2$	Relativistic invariance

Note that in case of TQFTs, we do not have interesting time evolution — $H = 0$. Although, we do have interesting *topological evolution* — the topology of the in-boundary and out-boundary can differ.

2.1 Example: Quantum Mechanics

This subsection follows the lecture notes of Pavel Mnev [Mne25].

Quantum mechanics may be viewed as a 1-dimensional (non-topological) quantum field theory. In this case

1. collections of oriented points — space.
2. oriented 1-manifolds equipped with a Riemannian metric — spacetime.

We define the theory Z as follows:

1. $Z(pt^+) = \mathcal{H}$ — the Hilbert space of the theory. On the other hand, $Z(pt^-) = \mathcal{H}^\vee$ — the dual Hilbert space.

2. Let $[t_0, t_0 + t]$ be the oriented interval of length t. Then $\partial([t_0, t_0 + t]) \simeq pt^+ \coprod pt^-$. Consequently, $Z([t_0, t_0 + t])$ is a linear operator, denoted

$$Z_t : \mathcal{H} \to \mathcal{H}$$

called the time evolution operator.

3. It follows from the axioms that $Z_{t_1+t_2} = Z_{t_2} \circ Z_{t_1}$.

Since the theory is not topological, we have an additional requirement

$$Z_\epsilon \sim id_{\mathcal{H}} + A\epsilon + \mathcal{O}(\epsilon^2)$$

for $\epsilon \to 0$. Up to a normalization, the operator $A \in \mathrm{End}(\mathcal{H})$ is the Hamiltonian of the theory.

Using the series expansion

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

for $n >> 1$, the time evolution operator takes the following form

$$Z_t = (Z_{\frac{t}{N}})^N = \lim_{N\to\infty} (id_{\mathcal{H}} + A\frac{t}{N})^N = \exp At$$

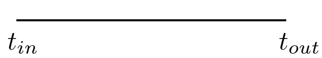
Remark 2.11. The axioms do not impose that the Hamiltonian A is a bounded operator. At the moment, it is any linear operator on $\mathcal{H}$.

Unitarity is not given by the axioms either. We have to impose additional conditions on Z to obtain a unitary theory.

2.1.1 Point Observables

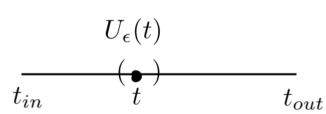
The discussion below generalises to quantum field theories in higher dimensions, but we focus only on quantum mechanics. The idea is to employ a *state-operator correspondence* and think of operators in terms of states — replace a point observable by a state on the boundary of a small neighbourhood of that point.

Consider a cobordism

Y = 

Let $t_{in} < t < t_{out}$ be a point in Y. We would like to define an observable supported at t. Consider a small neighbourhood of the point t

$$U_\epsilon(t) = [t - \epsilon, t + \epsilon] \subset [t_{in}, t_{out}]$$



The boundary of the neighbourhood is

$$\partial U_\epsilon(t) = pt^- \coprod pt^+$$

Let

$$\mathcal{O}_{t,\epsilon} \in Z(\partial U_\epsilon(t)) = \mathrm{End}(\mathcal{H})$$

be an operator on the Hilbert space.

Definition 2.12. A quantum observable supported at t is a family of elements

$$\mathcal{O}_{t,\epsilon} \in Z(\partial U_\epsilon(t)) = \mathrm{End}(\mathcal{H})$$

parametrised by ϵ, such that the limit

$$\langle O_t \rangle \coloneqq \lim_{\epsilon \to 0} Z([t+\epsilon, t_{out}]) \circ U_\epsilon(t) \circ [t_{in}, t-\epsilon])$$

exists.

Note that $\langle O_t \rangle \in \mathrm{End}(\mathcal{H})$, is an operator in the Hilbert space (state space) of the theory.

In a similar manner, one can consider several point observables, $\mathcal{O}_1, \ldots, \mathcal{O}_n$, supported at $t_1 < \cdots < t_n$ respectively.

2.1.2 Free particle on a circle

Consider the case of a free particle moving on a circle X of length L. In this example

1. The Hilbert space is $\mathcal{H} = L^2(X)$, square integrable functions on X.
2. The hamiltonian $\hat{H} = -\frac{1}{4\pi}\partial_x^2$.
3. Time evolution operator

$$Z([t_0, t_0 + t]) = \exp(-\sqrt{-1}\hat{H}t) : \mathcal{H} \to \mathcal{H}$$

For any point $x \in X$, there is a state $\delta(x) \in L^2(X)$ that corresponds to the particle being spatially localised at x. This state is denoted by $|x\rangle$. Given two such states $|x_0\rangle$ and $|x_1\rangle$, there is a finite probability that the particle in state $|x_0\rangle$ transitions to the state $|x_1\rangle$ in time t.

Definition 2.13. Let $|\psi_0\rangle, |\psi_1\rangle \in \mathcal{H}$ be two states in the Hilbert space. The transition probability amplitude from the state $|\psi_0\rangle$ to the state $|\psi_1\rangle$ in time t is the matrix element given by

$$K_t(\psi_0, \psi_1) \coloneqq \langle \psi_1 | \exp -\sqrt{-1}\hat{H}t | \psi_0 \rangle$$

Lemma 2.14. *The probability amplitude for a transition from state $|x_0\rangle$ to a state $|x_1\rangle$ in time t is*

$$K_t(x_0, x_1) = \sum_{n=-\infty}^{\infty} (\sqrt{-1}t)^{1/2} \exp(\sqrt{-1}\pi \frac{(x_1 - x_0 + nL)^2}{t})$$

Proof. Exercise. □

The probability of such a transition is the square of the modulus of the amplitude. The function $K_t(x_0, x_1)$ is called the propagator of the theory.

It follows from the axioms that

$$Z(S^1_t) = \mathrm{tr}_{\mathcal{H}} \exp(-1\sqrt{-1}\hat{H}t) = \int_X dx K_t(x,x)$$

Using the expression for the propagator, we find that

$$Z(S^1_t) = L(\sqrt{-1}t)^{-1/2} \sum_{n=-\infty}^{\infty} \exp\left(\sqrt{-1}\pi \frac{L^2 n^2}{t}\right)$$

On the other hand, the trace can also be computed using the eigenbasis of the hamiltonian $\{\exp\left(2\pi\sqrt{-1}x\frac{m}{L}\right)\}_{m\in\mathbb{Z}}$

$$Z(S^1_t) = \sum_{m=-\infty}^{\infty} \exp\left(-\sqrt{-1}\pi t \frac{m^2}{L^2}\right)$$

It must follow that

$$L(\sqrt{-1}t)^{-1/2} \sum_{n=-\infty}^{\infty} \exp\left(\sqrt{-1}\pi \frac{L^2 n^2}{t}\right) = \sum_{m=-\infty}^{\infty} \exp\left(-\sqrt{-1}\pi t \frac{m^2}{L^2}\right)$$

denote $\lambda = \frac{L^2}{\sqrt{-1}t}$, then

$$\sum_{n=-\infty}^{\infty} \exp\left(-\pi\lambda n^2\right) = \lambda^{-1/2} \sum_{m=-\infty}^{\infty} \exp\left(-\pi\frac{m^2}{\lambda}\right)$$

denote $\zeta(\lambda) = \sum_{n=-\infty}^{\infty} \exp\left(-\pi\lambda n^2\right)$, the partition function of the theory, then

$$\zeta(\lambda) = \lambda^{-1/2}\zeta(1/\lambda).$$

This is an instance of "T-duality", a duality of the theory under the inversion of the radius of X: $\lambda \to \lambda^{-1}$.

3 Chern-Simons Lagrangian field theory

We review aspects of Chern-Simons theory from a Lagrangian field theory point of view. Roughly, we need to provide an oriented 3-manifold and a simple Lie group to define the Chern-Simons theory.

Let M be an oriented 3-manifold, $E \simeq M \times \mathbb{C}^N$ be a trivial vector bundle over M. Let h be a hermitian form on E. In other words, h gives a symmetric sesquilinear form on each fiber $E_m \simeq \mathbb{C}^N$, varying smoothly with $m \in M$. Consider the set of connections on E

$$\mathcal{C} = \{\nabla : E \to E \otimes \Omega^1(M) \mid \nabla_{\partial_i} h(s_1, s_2) = h(\nabla_{\partial_i} s_1, s_2) + h(s_1, \nabla_{\partial_i} s_2)\}$$

Notation 3.1. The group of automorphisms of the pair (E, h) is denoted $U(E, h)$ — the Lie group of bundle isomorphisms $E \xrightarrow{g} E$ that restrict to unitary transformations on each fiber $(E_m, h_m) \xrightarrow{g_m} (E_m, h_m)$.

Let $\mathrm{Lie}(U(E, h)) = \mathfrak{u}(E, h)$ be the associated Lie algebra, i.e bundle endomorphisms $E \xrightarrow{\phi} E$, which restrict to skew-hermitian transformations on each fiber $(E_m, h_m) \xrightarrow{\phi_m} (E_m, h_m)$.

It is a general fact that the space of connections on a vector bundle F forms an affine space over the linear space of 1-forms valued in the endomorphisms of F

1. If A is a 1-form valued in the endomorphisms of F, then $\nabla + F$ is a connection on F.
2. Given two connections ∇ and ∇' on F, $\nabla + \nabla'$ is not a connection on F, but $\nabla - \nabla'$ is a 1-form valued in the endomorphisms of F.

In particular, $\mathcal{C}$ is an affine space over $\Omega^1(M, \mathfrak{u}(E, h))$. The fact that E is a trivial bundle implies that for every $\nabla \in \mathcal{C}$,

$$\nabla = d + A_\nabla$$

for some $A_\nabla \in \Omega^1(M, \mathfrak{u}(E, h))$. We draw attention to the fact that the above equation is always true locally on M, but in this case, the 1-form A_∇ exists globally.

The group of automorphisms of (E, h) acts on $\mathcal{C}$ via conjugation. Let $g \in U(E, h)$ and $\nabla \in \mathcal{C}$, then the gauge transformation of ∇ under g is

$$g.\nabla \coloneqq g \circ \nabla \circ g^{-1}.$$

This gives us the following gauge transformation law.

Lemma 3.2. *Let $g \in U(E, h)$ be an automorphism of (E, h), $\nabla \in \mathcal{C}$ be a connection. Then $\nabla = d + A_\nabla$*

$$g.A_\nabla = gA_\nabla g^{-1} + gd(g^{-1})$$

Proof. Exercise. □

Remark 3.3. The automorphism $g \in U(E, h)$ may be viewed as a smooth map $g \in \mathrm{Map}(M, U(N))$. The element $gd(g^{-1})$ is a Maurer-Cartan element,

$$gd(g^{-1}) = g^*(\nu)$$

where $\nu \in \Omega^1(U(N), \mathfrak{u}(N))$. (Will add a discussion on this later)

The action functional of Chern-Simons theory is a function on $\mathcal{C}$.

Definition 3.4. The Chern-Simons action functional $S_{CS} \in \Omega^0(\mathcal{C})$ is a function on the space of connections $\mathcal{C}$, given by

$$S_{CS}(d+A) \coloneqq \int_M \mathrm{Tr}(AdA + \frac{2}{3}AAA)$$

for every $d + A \in \mathcal{C}$.

Now we show that the functional S_{CS} is in fact quite a natural function on $\mathcal{C}$. Let $\nabla \in \mathcal{C}$ be a connection on (E, h). The curvature of ∇ is given by

$$F^\nabla = \nabla \circ \nabla = dA_\nabla + AA$$

Lemma 3.5. *Let $\nabla \in \mathcal{C}$ be a connection. The tangent space to $\mathcal{C}$, at ∇*

$$T_\nabla \mathcal{C} \simeq \Omega^1(M, \mathfrak{u}(E, h))$$

Proof. Exercise. (Hint: $T_\nabla\mathcal{C}$ gives first order deformations of ∇; or tangent spaces to vectors in a vector space) □

Definition 3.6. Let $\eta \in \Omega^1(\mathcal{C})$ be a canonical 1-form on $\mathcal{C}$, defined by

$$\eta_\nabla(\delta A) \coloneqq \int_M \mathrm{Tr}(F^\nabla \delta A)$$

for every connection $\nabla \in \mathcal{C}$, every $\delta A \in T_\nabla\mathcal{C} \simeq \Omega^1(M, \mathfrak{u}(E, h))$.

References

[Mne25] Pavel Mnev. Lecture notes on conformal field theory. *arXiv preprint arXiv:2501.06616*, 2025.